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These materials were written with the aim of reflecting the thinking of Cambridge Conference on School Mathematics (CCSM) regarding the goals and objectives for school mathematics K-6. In view of the experiences of other curriculum groups and of the general discussions since 1963, the present report initiates the next step in evolving the "Goals". Three areas considered in this report are geometry, functions in preparation for calculus, and applications. Two working papers are presented on applications - probability and mechanics and slopes. One working paper on circular functions is included. Fifteen working papers are presented involving geometry and geometrical concepts. The papers on geometry include examination and description of common objectives; playing with figures, blocks, and tessellations; constructions; graphs and polygons; tessellations; dissection of figures; order; measurement; similarity and map making; symmetry, congruence, and rigid motion; transformation groups; rotations and matrices; iterated reflections in mirrors; knots; and spheres, cylinders, and torus. [Not available in hard copy due to marginal legibility of original document]. (RP)

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CAMBRIDGE CONFERENCE ON SCHOOL MATHEMATICS

REPORTS OF

1965 SUMMER STUDY

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PROVISIONAL APPROACHES TO GOALS FOR SCHOOL MATHEMATICS

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INTRODUCTION

The Cambridge Conference on School Mathematics during the summer of 1963 invited people with a research interest in mathematics (though not only mathematicians) to consider a long range problem. They were asked to answer, as best they could, the question:

> "What mathematics would you like to have students learn from kindergarten through twelfth grade, if there were no restrictions other than the child's innate capacity?"

Difficulties, such as the need for highly trained teachers, were to be ignored on the ground that, if the program was meritorious enough, these other problems could be solved in the long run.

The much discussed report "Goals for School Mathematics" (Houghton-Mifflin, 1963) was the conferees' response to their assignment. Although it raises general pedagogical questions (the discovery technique, the spiral approach) and makes contact with some experimental material tested in the classroom, it is largely in the framework of mathematical needs rather than classroom needs. It gives opinions on what it may be useful for the student to know at a certain level, but does very little to suggest the specific unit which may teach such material.

Since that time several people connected with the conference have worked in the classroom to develop a practical response to some of the challenges of the "Goals" report. From 1963 to 1965 the Miss Mason School (Princeton, New Jersey) under the initial direction of the late Wilks developed detailed units for pre-first graders that included number line and other concepts recommended by "Goals". B. Friedman worked out and tried a geometry based on mirror symmetry for Junior High School students in Berkeley. At Estabrook Elementary School (Lexington, Massachusetts) E.Lomon with several of the local teachers tried material connected with inequalities, real number, symmetry, probability and slope in grades 1 to 6. In the summer of 1964 at the Morse School (Cambridge), under the auspices of CCSM, a team of research people and school teachers under the chairmanship of A. Gleason tried out subject matter from counting through symmetry to number theory, for a five-week period. Several of these units were tried again in schools during 1964-65. A report of those projects will soon be available.

To some of those in CCSM it seemed to be time, in view of this experience, the experience of other curriculum groups, and of the general discussion since 1963, to take the next step in evolving the "Goals". For this purpose CCSM invited the present group to meet in June and July of 1965. The group consists largely of mathematicians, some of whom had curriculum development experience, and some who had not. We were charged with the task of producing material for the experimental classroom, in geometry and applications for grades K to 6. We were to show how one could make a start on translating the mathematical needs expressed by the "Goals" report, to meet the needs of the young student in the classroom. The importance of the "function" concept to applications led us to treat this as a separate problem.

Fortunately for the present writers, we were not required at this stage to produce units for the usual classroom and its teacher. The present level of experience argues against the success of any such attempt and we feel none of our material should be used in this way. Rather we have addressed ourselves to those individuals and groups with experience and capability in the development of experimental mathematics and science curriculum materials for the schools. We hope that we have expressed ourselves in sufficient detail, and clearly enough, so that such people will know what type of classroom experience we have in mind. Then if a researcher wants to test our ideas, he can do so in his own way and yet feed back information relevent to the general CCSM enquiry as related to the "Goals for School Mathematics".

We are aware that we have fallen short of even this limited task; that there are fuzzy aspects as to how some concepts are actually to be taught, and incomplete development along some lines. We are also aware that some of the material is likely to be too difficult for the young student, at least in the garb we have given it. We have offered such material in the belief that the child's innate abilities have still to be suffic ently tested, and in the hopes that more teachable variants would be found by those who choose to read this report and work with some its suggestions.

Many of the things we discuss here have already been suggested and worked with by other individuals and groups. We include them so that they can be viewed in the "Goals" context and so that future research with such material may in part be made to bear on questions in that context. The bibliography at the end is not complete, but we hope that it gives some idea of the debt we owe to previous writing and research.

We hope the reader will bear in mind the general principles and outlines of "Goals for School Mathematics" while reading this report. While sometimes, in the papers that follow, explicit reference is made to pedagogical technique or educational goals that are to guide the teaching of the material, often these comments are absent in the expectation that the "Goals" report has given the necessary direction and context. If the reader still has the patience with all the burdens we have placed on him, we hope he will read on and find some suggestions and methods useful to his own curriculum and classroom research.

GEOMETRY

We have found that the problem of designing a curriculum for the study of elementary geometry in grades K - 6 does not have an obvious and straight forward solution. There are no objective criteria for evaluating a given curriculum or for the comparison of two curricula. The production of such criteria requires a good deal of concrete classroom experience with such subject matter at the various grade levels. Unfortunately, sufficient experience of this type is not at hand, and it is not known what children of a given age can digest and profit from when suitable preparations have been made in earlier grades. Moreover, it is clear that any forward-looking explicit statement of a curriculum serves merely as hopeful anticipation as to what can be accomplished. In the final analysis, the decisive factor (unless there exist teaching materials which do not require a teacher) is the teacher in the classroom - his mathematical proficiency, his teaching skill and technique, his enthusiasm, and his understanding of and empathy for children. The training and preparation of teachers of mathematics has not entered into our considerations, especially since this problem is to be the focus of a conference to be held in the summer of 1966. Rather, our concern has centered on the mathematics itself - its content, the approach to it, and its organization.

By and large, we have found ourselves in substantial agreement as to what the "geometry experience" of a child in grades K - 6 ought to be, which facts he should become acquainted with, and what attitude and intuitions he should develop. For example, the principles guiding the development of teaching style and materials should include: the spiral approach - where topics are repeated and then extended at different stages, with increasing levels of sophistication and deepening of the child's understanding; the discovery method - in which children learn things from their own experience and by indirection whenever possible; the open-ended approach - where topics are to be examined and investigated and not tied up into neat packages. It follows from this that, especially in the earlier grades, the approach to geometry should be concrete, manipulative and operational; that is to say, the child should learn by "playing" with geometric objects. Formal proofs are to be avoided (while the student's ability to reason deductively is to be stimulated) and the passage from the concrete to the abstract should be slow, tentative and imperceptible. Among the objectives of all this work, as we see them, are the building of a solid intuition for Euclidean space,the preparation of the child for an axiomatic treatment of geometry, and the fostering of familiarity with approximations and limits.

We have chosen to take these guiding principles for granted and, therefore to structure our discussion according to the internal logic of the mathematics involved. Thus we have constructed first a general framework for geometry - on which pieces of geometric information can be hung and organized. This may be viewed as a curriculum generator in the sense, that various curricula can be constructed out of it according to the needs and desires of a given school. In particular, this framework deals with more material than can be covered in a single curriculum.

This material is then broken down into various topics (the list of topics is meant to be suggestive or illustrative rather than exhaustive) which are treated with varying degrees of detail. Some topics are discussed in profuse detail - even to the extent of describing what should go on in the classroom - while others are sketched in the form of a developmental sequence. Other topics are omitted entirely. Of course, the selection of topics for extensive treatment is, in part, arbitrary, and is not necessarily related wit the importance of these topics and the emphasis to be placed upon them. On the other hand, each topic is designed according to increasing order of complexity and difficulty with no hard and fast decomposition as to the grade level at which its components should be taught. The decision as to exactly what should be taught, and when, is best left to the ultimate users.

FIRST PROVISIONAL WORKING SUMMARY OF GEOMETRY

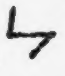
Geometry for a child begins with his first perceptions of physical objects. As his hand and eye muscles develop, his simian curiosity and his appetite for exploration lead to his exercising those muscles and with them his brain circuitry and nervous system. From the interplay of these he develops, among other things, his world of geometry. In this world three levels can be distinguished for our present purposes: physical objects, such as bricks, balls, boards, paper, wire, etc. occupying a three-dimensional (3D) world; drawings, or pictorial models, of physical objects, or of certain aspects of them, occupying a 2D or 3D "drawing (or model) world"; and such abstractions as line, plane, circle, etc., occupying a 3D abstract mathematical world. We shift and slide between these levels, sometimes even mixing them up by thinking, for example, of a mathematical circle as being in the room with us, or of a piece of cardboard as being a drawing.

We shall therefore use figures in this discussion to mean either physical objects, drawings, or abstractions, depending on which level it is in which we are working. We are also concerned with mappings, or transformations. For physical objects, a mapping is a movement of the object or of a part of it; moving a car, folding a piece of paper. For a drawing it can be an actual physical movement of the material on which the drawing is made, or only an imagined but describable movement. For an abstraction it is a function in the mathematical sense of that word. In geometry for grades K-6 we shall emphasize figures and treat mappings only secondarily.

The objects children begin with are those near them in the physical world; pots and pans, balls, oranges, spoons, sticks, stones, blocks, etc. Through pre-school manipulation and exploration children have already begun to investigate, quite naturally and entirely on their own, certain ideas and approaches which run through or too much of mathematics and indeed much of the rest of the world. They learn aspects of a single figure; edges and corners, roundness, straightness, holes, and perhaps some symmetry. They also compare two figures: does one fit with or inside the other, do they look and act alike? And they make new figures from old: putting one object inside or next to another is using two figures to make a third; tearing a piece of bread into pieces is making new figures from old; and so is stamping a rubber stamp or folding a piece of paper or cloth.

These are all natural processes, and it is these that underlie all treatments of figures.

In learning figures children begin with physical objects and eventually are led to drawings as substitutes, substitutes which take on a life of their own. Slowly a "zoo" is built up, and classifying begun; and the children learn the figures and their properties by seeing and handling them and by finding them and by making them, whether physical objects, or physical objects approximating drawings (as a stretched thread is almost a line segment), or drawings. They also manipulate them: comparing, matching, taking apart, composing, associating, the more freely the better. All these should be kept in mind in constructing any geometry program for children.

The figures themselves not only divide into the three groups mentioned above, but also can be distinguished by dimensions. Although all figures will be regarded as being in 3D space, each such figure has an intrinsic dimension and an extrinsic one. A line segment has intrinsic dimension 1 no matter how it is bent, or not bent, in 3D space, but its extrinsic dimension does depend on how it is bent (or not bent). If a pipe cleaner is straight its extrinsic dimension is 1, and we shall say it is "1D in 1D". Bent in one place like this < , it is "1D in 2D", because it can be laid flat but not on a line; and bent at two places so that it sits up on a table like this,  , it is "1D in 3D" since it cannot be put flat on the table top. The surface of a ball is 2D in 3D, a triangle is 1D in 2D, a disc is 2D in 2D.

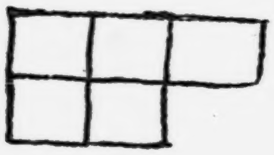
In building the zoo, we begin with the most familiarly shaped physical objects: blocks, bricks, balls, wedges, etc., as solid, or 3D in 3D, objects: paper and plastic figures are "almost 2D' objects, and string, wire, thread, and pins are almost 1D objects. Aspects of these are end and corner points, edges, faces, insides, outsides, and so forth. These can be straight or flat or curved or rounded, or a combination of these qualities. The transition to drawings can start with slavish tracing of these objects and then moves to more general drawings, done by designed and composed tracings, and finally to freehand. As a child progresses through the grades the zoo expands; in the class of physical objects, the types of drawings and the sophistication of drawing methods, and the class of such abstractions as line, ray, plane, circle, polygon, half-plane, etc.

A great deal of elementary geometry falls into the categories loosely titled related figures and forming new figures from old. Such figure relations as inclusion, congruence, similarity, perpendicularity, and parallelism, are standard, while others such as equidecomposability, tangency, equicomplementarity, and interiority are less so. There are also such relations as "being the convex cover of", that is, being the smallest convex set containing or "being the flat cover of" or "the boundary of" or "the interior of", each of these being a relation that is a figure function. Here we propose to only discuss the five standard ones listed above together with the figure functions "flat cover" and "convex cover". Forming new figures from one old figure can be done internally by decomposition (chopping, slicing, extraction) or externally either by mapping the figure or by observing a decomposition it forces on its complement. Given several old figures, new ones can be formed by adjunction, union, intersection, and cartesian multiplication. Almost all of these relations and methods are at least touched upon in any treatment of elementary geometry, and all will be considered here.

For these topics there are certain natural lines of investigation and development. Consider, for example, any figure relation and figure A. To find out about the class of figures so related to A, one first tries to find some figures that are in this class, or, failing that, to see if the class is void. If there are figures in the class, we try to find some criteria, not too difficult to use, that will tell us whether or not a figure is in the class. The rest of the job is to divide the class into whatever subclasses happen to interest us. For example, if the realtion is inclusion and A is a point, we can find line segments, discs, triangles, sets of three points each, blobs, all sorts of figures that contain A, and many that do not. If we decide we are curious about all the line segments that contain A, we can ask what all of these, as a class, are. What do those that have A as an end point look like? Among these, what are all those that are congruent to a given line segment, and what is the locus of their non-A end points? What are all the line segments that have A as an end point and go through a given second point B? What would their union be? What are all those that have A as an end point and their other end points on a given line? Among the latter which ones, if any, are perpendicular to the line?

Similar questions could be asked about discs containing A, or triangles, or square regions. Some answers might be dull, some interesting. We could change A to two points, or three points, or a line segment; we could change the relation to that of congruence. The many questions possible will vary in difficulty, interest, and generative power. The point here is this; given figure relations and figures, one can choose any relation and any figure and generate many questions about the figures having that relation to that figure, and that leading children to generate, out of their own curiosity and imagination, questions like these is a desirable way of their exercising and developing geometric curiosity, imagination, and knowledge.

There is somewhat the same freedom in building new figures from old. For example, in composing figures we are given a set of figures and certain ways of combining them. (In Kindergarten the figures are blocks or cutouts, the ways are unrestricted.) There are two inputs here: a set of figures, and a method of combining them; and the output is a figure. Given any two of these, one can ask for the third; for a very simple example - I have four pipe cleaners; how can I connect them on this table-top to make a tree? To make a tree with two limbs? Or: I have 5 congruent squares connected thus:



How can I fold them to make an open top carton? Here the set of figures and the resulting figure are specified, and part of the composing is partially restricted (the squares are connected); the rest of the composing is to be found. Or we can be given a figure, say a rectangular region, a method of combining (tiling), and be asked to find those plane polygonal regions each of which will tile the given figure. Or for example, here is a cylinder; the method of composition is cartesian multiplication: find two figures that so compose to form the cylinder.

There is a similar situation with decomposition. Let F be a figure, say a solid cone. How can it be chopped up or sliced or decomposed? Can it be cut up into balls? Solid cylinders? Can it be sliced into discs? Conical paper hats? Circles? Line segments? Points? Two congruent figures? Points and line segments? Obviously there are many possiblities.

DEVELOPMENT

Figures

Play with blocks and other figures in the list of physical objects; on these identify faces, edges, corner points. Discussion of roundness, straightness, and flatness. Sorting of figures. Ball, sphere, brick, box disc, circle, line segment, three and four-sided plane figures, rectangles, squares, points; equilateral, and right triangles; tetrahedra and polyhedra, both surface and solid. These figures found, then made from paper, cardboard, clay; for example, folded paper to make straight edges, points, and square corners. Polygons, closed polygons, regular polygons, simple closed plane curves.

A given line segment is part of many line segments; taken all together, these make what is called a straight line; similarly two ways are determined by the given segment. Any triangular region is included in many other such regions; the union of these forms what is called a plane. Angle can begin crudely as the trace of a corner of a plane region and become the union of two line segments with common and point; right angle starts with a square corner and later is either part of any bisection of any straight angle.

Cylinder, prism, cone, pyramid, wedge, and torus, and their solids, start as named physical surfaces and blocks, and later are viewed as the unions of collections of lower dimensional figures, and as cartesian products where that is possible. The conic sections appear as such as also as loci. Annuli and annular regions, Moebius strips, tream, knote; composed figures.

Figure relations

Develop from the beginning the idea of one figure being contained in, or part of, another figure. Faces and edges are parts of blocks, a squiggle on a face is part of the face and part of the block. There are many line segments and many points contained in any line segment. A disc contains many discs, line segments, square regions, ... and is contained in many discs, rectangular regions... Determining all figures of a particular kind that contain (are contained in) a particular figure; later raise maximality questions about these figures. Ordering by inclusion should begin early, be not confined to similar figures, and should progress from two figures to finitely many to nested squences of figures. Intersections of pairs of 1D and 2D figures, physical and drawn. Intersection of line and plans, plane and ball, etc. The intersection of a decreasing sequence of line segments, of rays. Intersection of a finite number of half-planes, half-spaces.

Given any figure, by drawing all line segments on point pairs in the figure lead into the smallest convex set containing the figure; replacing line segments by lines introduces the smallest flat set containing the figure. Find these smallest sets for certain cases, including figures of two points, three points, and four points. Given a convex set, what of it can we erase and yet recover the set; extreme points for discs, balls, tetrahedra, polygonal plane regions, mixtures.

Congruence begins as the motion of two physical objects fitting precisely in superposition, translates into the idea of two drawings that can be fitted together precisely, and ends with there being a rigid motion that maps one onto the other. Finding, then making, congruent pairs of physical objects (straight edges, folded paper, cut-outs); then, given a physical object, to find or make a physical object or drawing that is congruent to it; then, given a drawing, to make a congruent drawing, for line segments, right angles, circles, angles, polygons. Always begin with free copying of figure, later narrowing to restricted cases. Bisection problems, first using string and paper folding, for line segments, angles, discs, rectangles, some triangles.

Perpendicularity can begin with the edges of a paper square corner; tracing in a plane leads to perpendicular line segments and then to perpendicular rays and lines. Two coplanar segments, rays, or lines are parallel if they have a common perpendicular. Construction of perpendiculars and parallels, unrestricted or through a given point, using square corners and then straight edge and compass. Using blocks and paper square corners, introduce idea of a plane and a line being perpendicular. Follow with parallelism for two planes, a plane and a line, and two lines. Look at relations between parallelism with the property of either coinciding or not intersecting. Perpendicularity for two planes.

Similarity.

Composition and decomposition of figures

Composition of figures should start with free composition of physical objects, including 3D, 2D, 1D and 2D-in-3D figures; then ask for certain figures to be composed. Later have free composition of drawings and free hand drawings. In the plane make polygons, curves, trees; compose line segments to make such standard figures as isosceles equilateral, and right triangles, squares, rectangles, parallelograms, etc.

Adjunction of line segments to make a line segment, of angles to make an angle, of some polygons to make a polygon. Tiling of line segments on a line segment, of angles in an angle, of polygonal regions in 2D regions; the integral number line and the integral number plane. Tiling 3D regions within a 3D region. Folding 2D cardboard figures to make 2D-in-3D figures: first free, then prescribed, e.g., four congruent equilateral triangular regions to make a tetrahedral surface, or five congruent square regions to make an open box. Given enfigured cardboard, to cut and fold to get prescribed 2D-in-3D figure.

Do much decomposition of figures into subfigures: first free, than restricted; use string and paper, then drawings. Example: here is a rectangular region - can you fold or divide it into two pieces that fit each other exactly? In how many different ways can you so divide it? Decomposition of a particular figure into particular subfigures. Folding of a paper triangle to show that the angles of a triangle can be joined to form a straight angle. Decomposition of line segments into points, rectangular regions into congruent line segments and into points, discs into circles, solid cylinder into congruent discs piled along a line segment and into congruent line segments attached over a disc, etc. Introduction of cartesian product of two figures. Analysis of certain figures to see if they are cartesian products. Decompositions of cone, pyramid, wedge, ball, etc. into line segments 1D-in-2D, and 2D-in-2D subfigures. Start coordinatizing, by length, on line segment and circle, then on cartesian product sets in 2D and 3D; cylindrical and polar coordinates. Simple equidecomposability problems.

Begin separation phenomena with "inside" and "outside" of rope circle, stick frame, cardboard carton and sphere. A simple closed curve in a plane separates the plane, and so does a line in the plane, but differently; similarly for a plane and a simple closed surface in space. Observe that 2 points can be separately enclosed by: line segments in their line, discs in any plane containing them, and balls in space. Intersection of half-planes and of half-spaces.

Section I - Chapter 2

FUNCTIONS IN PREPARATION FOR CALCULUS

A Point of View

If there is something that distinguishes the point of view of this working paper from others being written at the conference, it is contained in the assumption that children can imagine things they can't touch or even see. It is not a simple matter to make such a characteristic come clear and be explicit, because probably it really isn't explicit at all but rather implicit in the way one would write and teach the mathematics itself. To begin with the objects of this particular discussion are functions, and when you try to look at a function it becomes shapeless, or falls into fragments, or hides behind a name, or assumes some spurious form or other. Worst of all, it finds a haven in the bosom of the sophisticate who doesn't see anything especially hard about a function. But just try to dissect and examine a function that is at all non-trivial with only the resources of the unsophisticate, say a K-6 child, or your lawyer, or your wife, or your engineering friend even, and you'll quickly reach an unrewarding point of diminishing (ed) return.

It may help in getting a discussion started to assume that everybody is hopeless in this regard except the sophisticate who knows more about functions than we do, and just perhaps the K - 6 child, too, who still has the time as well as the curiosity to entertain, and even ask some sensible questions in a context (meaning learning situation) consisting of the universe of functions. Furthermore, the K - 6 child can be forced to listen to some information and learn it, he'll question without quarreling, he'll wander off and doze from inattention and fatigue rather than from prejudice and disdain, and he'll command a lot more sympathy and worry from his teacher than anybody else does. Functions are the ingredients of calculus so we must take advantage of the little kids' wide open eyes and ears, their trusting attitude, and subtly or overtly (as suits us) bend their thinking to our will.

The study of a particular function may be considered in two useful aspects, global and local. Some global properties are size (cardinality of the domain set), periodicity, existence of an inverse (one to one-ness). Local properties at the beginning are limited to single domain entries, or perhaps to small groups of them; in the story function "Snow White" has a happy ending. Until children have enough number work to know about fractions, or get a feel for the number line as a continuum so they have dense sets to use for their domains, there's little point in belaboring local phenomena early. But even when measuring with a ruler marked in inches, say, it could be noticed that something might happen near the middle of the ruler contrasted with what might happen way out at the ends. The idea of a neighborhood, thus topology of a domain, ought to come in as soon as it makes sense.

To reinforce the function as an entity, it shouldn't be difficult (returning to the global view) to compare functions, classify them, find sub-functions, compose them, and, as soon as some structure is found in the domain and range, produce an algebra of functions. Numerical-valued functions produce examples of these things most readily and are ultimately especially useful for elementary analysis. Furthermore, practice with arithmetic, that,is, the "facts", familiarity with fractions, decimals, etc., can be hung on a framework of the study of functions. Simple counting is a successor function, addition is a function on pairs, numeral is a function on numbers to names, equations are functions on numbers to true, false. This is not to say that everything is a function, but that there must be continuing exercise in the recognition of those things that can be viewed as functions for some use or just for fun.

Still in the global view, suppose that children can add and multiply whole numbers, and that there are ten kids in a particular group. Joined in a circle they count around and each child collects the numbers that he himself says. Then they pair off and add each others' numbers, each time asking who's got the answer. Each pair of children will find a "sum" child who goes with them, and the homomorphism this game generates can give exercise in addition of numbers to begin with at an early stage, and ultimately, at a later stage, a formulation of the Euclidean algorithm and use of residue classes of integers.

Pictures are essential in all function study, and included as pictures are tables with connecting arrows, graphs, facsimilies of physical setups, equations - anything by way of a record which collects data about a function and optimally suggests extra properties of it. It should emerge in the student's mind, over any sequence of lessons and over the years representing arms of the learning spiral, that a function is a mental construct that is only represented in part by any "picture" and can be attacked definitively only by mathematical reasoning. Or better, by reasoning alone, which reasoning can be called mathematical when and if it produces definitive results about the function. For small finite functions, the exhaustion of all possible cases is mathematical reasoning. For large finite and infinite functions the mathematical reasoning assumes its essentially distinctive character, and by this we mean relies on the techniques of logical qualification. By sixth grade, we might presume, most normal children should begin to sense this and, hopefully, be intrigued by it. The interplay between plausibility with persistent skepticism and rigor with finality, recognizing all the time the use and peril of both (how they on the one hand reinforce each other, and on the other hand get in each others' roads) is the main object of the study of functions, from the beginning of a curriculum. Within mathematics the narrow use of functions as a foundation for calculus is secondary, but, in the present instance, controls the direction of the exercises.

FUNCTIONS IN K - 6 IN PREPARATION FOR CALCULUS

Some Suggested Directions

We begin with the hypothesis that functions should be shown to children whenever the opportunity arises, and that there are such opportunities at every stage of their school experience beginning in kindergarten. In class periods devoted to mathematics explicitly, examples of functions should be observed along with examples of geometric things and numbers. Outside of mathematics lessons as such, functions and the terminology associated with them occur frequently, are widely applicable in virtually all subjects, and should be built into the vocabulary and expressions children are taught to accumulate and use. The concept of a function is of course important in its own right, it is of use more broadly than just in mathematics, and as a device for teaching it makes use of all the objects that arise in mathematical work for its illustration. As correspondences between sets of things are observed and practiced with, these sets of things and their members gain structure and life especially as techniques and terminology are developed to treat both the structure of these sets internally and their relations with each other.

It is the purpose of this brief essay to suggest ways by which functions may be brought into the experience of children in the grades K - 6, and also discovered in and extracted from experiences they already have or are having in other activities within or outside of school. This may be very unstructured and undirected to begin with, but by grade 6 ought to be strongly influenced by and pointed toward the theory of functions as mathematicians know it and use it. This is not to say that the function idea is an absolute trunk of the tree of mathematics from which all else branches out, or that all else in the mathematics curriculum has to be connected with or related to it in detail, but such a metaphor as this might aptly consider function to be the life blood of the tree of mathematics. We make this remark to hedge against the accusation that we view the introduction of functions at the outset as a cure-all for the ills of mathematics curricula. We do not, any more than should we view a preoccupation with sets or with binary numerals as such a panacea. But functions are fundamental in mathematics in its broadest definition, simple things about them are understandable by young children, they come in a wide variety, and they are interesting.

If children are to be expected to buy this package - a study of functions interwoven into their general studies - we better lay in a store of aspects of them which are interesting in an open-ended way, that is, ultimately fascinating. What these are can be discovered only by experience in teaching them, but one way to begin is to go out on a limb and consider some characteristics of functions that interest us and which persist in doing so the more intensively we investigate them.

Traditionally, functions have been introduced one by one, the graph of each one plotted, extreme values of each one formed, problems posed for which a single function is useful in each case, and so on. Only much later, in fact never in the experience of most non-math majors, are classes of functions viewed as determined by properties of the functions themselves, functions as objects. Yet this latter point of view may be the easier one to start with for children, and we will explore this attack in what follows. Any non-trivial function is apt to be intricate enough to demand considerable skill with formulas (meaning mathematical symbollism quite generally) in order to study it, and so we propose that initially functions be talked about as objects roughly in the same way squares and circles and numbers are. To lead toward this and support it, then, work in geometry, arithmetic, and algebra should include emphasis on linear order, nesting of neighborhoods, and other things appropriate for the later study of local properties of particular functions, when problems in analysis are studied with greater precision and in depth.

As targets beyond the K - 6 range, one might consider what properties of functions, properties of their domains, and what devices for examining them, ought to be built into the experience of students in order that they can tackle an honest treatment of continuity of a function on the one hand, say, and on the other hand a vector space of continuous functions. The reader can make a list of such ingredients for himself, it is not brief; but we believe he will discover that many of these elements in the study of function are intuitive enough to children to be discussable with them in the K - 6 ages. As we noted above, the overture must play on the theme, a function is a function is a function, and let the resolving power of instruments invented to examine particular functions increase at a somewhat deliberate pace.

Children learn new words every day. Why not learn a new function every day, too? It will reenforce the notion of a function as an entity to be examined for its own characteristics and to be compared, contrasted with, and related to other functions. This is important, a function should emerge as an object. Each time a function is presented there should be enough discussion (or play) with it to identify its domain and range, the rule which describes what the correspondence inherent in the function is, and some kind of picture of it. These observations should become regular exercises; just as we would expect to do some reading, numbers, geometry, and other things practically every day, we would do some relating and corresponding, too. Let us insert image and pre-image, inverse function (one-to-one correspondence), composition of functions, and cartesian product of sets into the catalogue of things to be dealt with from the beginning, and see how these notions might be talked about in, say, kindergarten and first grade.

Ask each child in the class the name of the street on which he lives. If someone doesn't know, have him find out and report tomorrow.

Make a table everyone can see and connect the names with their corresponding streets. Likely more than one child lives on the same street. Some observations would be:

Are there more names than streets? (cardinality of sets)

Which column is "Joe" in? which is 25th in? (Domain and range sets)

How did we know "Elm" goes with "Eve"? Possible answers, "We asked her", or "You drew a line between them". (rule of correspondence)

Who lives on Broadway Street? (pre-image)

What shall we call this game? (name of a function)

Just a few observations of this kind are enough. The next day another game like this, when played, may generate other questions but gradually the kinds of things seen in such exercises should classify themselves into stock questions about functions that arise over and over again. These eventually come to define the nature of the entity called a function, but there is no hurry about it. A child may play with blocks for many years before the naming of a cube, "cube", adds more to his knowledge and mental versatility than it detracts by appearing to complicate a familiar and clearly simple configuration. Thus with functions, the indoctrination process must be gentle and seductive, the objects themselves entering by the side door, so to speak. Only much later on (by grade five, perhaps?) can they appear as actors on the main stage, listed on the program.

The simplest elements of a function are the existence of two sets, one of them favored over the other, and a decision process attached to each member of one of the sets; domain, range, and rule. The decision process is to be unambiguous. The example just cited contains the set of students, the set of streets, and the rule x lives on street y. It also contains a non-functional relation if the preferred set roles are reversed, and this appears when two children live on Elm Street, for instance, and it is required to specify the person who lives on Elm Street. The situation is different when there is doubt about what the answer should be. Certainty and doubt can distinguish between functional and non-functional relations.

Some functions which could be used for these early exercises are, to begin with, possessions of the children, that is, a common domain could be the set of children in the class. "Belonging to" is a familiar idea in the context of personal possession

child —> his birthday
child —> his coat
child —> his desk (where desks are distinguished in some way)
child —> color of his house

A function on a set to itself arises when choosing partners for projects done by couples. This example generates a function-making machine as partners are chosen in different ways. Thus

child —> his partner	Tuesday's choice
child —> his partner	Wednesday's choice

and these are different functions, Tuesday's choice ≠ Wednesday's choice. We remarked earlier that the assignment of a name to a particular function tends to emphasize the possibility of its existence, and the present example illustrates this.

At some place along the way, at least as soon as these matching up games become expected and familiar elements of the "activity space" of school, some generic name should be used to refer to them. "Function" is the only candidate, apparently. Our hesitation on this point stems from two misgivings - one the artificiality of injecting a stranger (the word) into a comfortable family (the games), the other the sheer ugliness of the word itself in both appearance and sound. But this has to be done just as soon as more is gained than lost by it. A tentative suggestion is to observe that in some games there is a doubt about the outcome,

Who lives on Elm Street? Either Susie or Joe, whom did you mean?

and in others there is no doubt at all,

Where does Joe live? On Elm Street, certainly.

In cases such as the second of these the teacher can let fall the utterance "function" each time, but take care to avoid its use otherwise.

As a preparation for the introduction of functional notation the children should get used to talking about the function games with names for the functions themselves, as we have been emphasizing, and along with this they should use some rudimentary form of "f(x)". This might start concurrently with reading and writing exercises, so that a written form of the record of the street game would include statements such as,

Joe's street is Elm,

which would later be written,

The street for Joe is Elm,

sooner or later to become abbreviated to

Street(Joe) = Elm.

Up to the time at which geometric objects and arithmetic are symbollized, then, children should be led to experiment with those aspects of functions described by domain, range, rule, image of a domain element, and to understand that a function (if only perceived as a particular kind of game) can be described in writing. We are not in a position to specify when this stage can be reached, but we are proposing that the children be made ready for, and be given the vocabulary to use in, exercises of the following kind using "numeral" and "place holder" symbols.

Numerical-valued functions have been encountered, presumably, before much arithmetic has been done. With arithmetic available in the range of such a function, exercises with functions can be extended to their algebra. An example should be sufficient to explain this here, many examples necessary to explain it to a class. The post office provides one of these. Consider all packages in the shape of boxes. The length function L assigns a number, which is the largest of the three dimensions (in the usual sense), to each package

$$L: \text{package} \rightarrow L(\text{package})$$

The girth function G picks out, for each package, its girth in a plane perpendicular to the axis of the dimension L(package).

$$G: \text{package} \rightarrow G(\text{package})$$

Since L(package) and G(package) are numbers, uniquely assigned to each package, their unique sum is assignable to each package and a new function is born. By the time this kind of game is familiar, the need for record keeping, that is, writing things down, might support the plausibiltty of the name L + G for this new function.

We do not address ourselves, in these remarks, to the construction of specific lesson plans for the teaching of the example in the last paragraph. We do plead, however, for the inclusion of an exercies of its kind just as soon as the function game is understood and playable, and right along with the study of addition in arithmetic. It (and many others like it)* should be presented to the children as soon as it can be made comprehensible to them for several reasons, among which we observe that

1. it does something with functions beyond viewing them just one by one,

2. it introduces a binary operation in a set of abstract (sic.) things,

3. it contains a reenforcing review of virtually all that has been said about function prior to its occurrence,

4. the record keeping associated with it focuses attention on the pairwise notational distinctions between x, f, and f(x), in whatever symbollic guise a teacher may prefer to write them,

5. it is the kind of process that can be repeated with variations of content, reemphasizing important ideas in a sugarcoated drill.

It can be decided only by trial in the classroom whether the algebra of functions is a more or less primitive notion than is their composition. Composition is the simpler in that is has fewer ingredients (no operation in the range is required) but it may not seem so to the beginner. However this may be, composition games should be played as soon as composable functions are at hand, and notation leading to $f \circ g(x) = f(g(x))$ worked into the record keeping apace. Examples are easy to find, for instance

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*Whenever measuring is done, a function (object ⟶ its measure) is in use. See "Circular Functions" for an example of how measurement can be exploited.

child → his street → on bus line, or not
child → his desk → number of pencils on it
package → length plus girth → amount of postage.

The first two are kindergarten examples; the last combines addition with composition in the same illustration.

We are influenced by aims more lofty than just keeping primary school children busy, and two of these serve to direct our thinking at this point. Mapping structures by isomorphism is one of these, and recognizing a topology in the domain of a function to support analysis (calculus) is the other. When the function concept is useable, and, in particular, when functions with inverses (one-to-one correspondences) are recognizable, their isomorphisms and homomorphisms should be accessible at least by example. This is where algebra in the early grades could start, and here is where a pre-algebra essay should be written. We are tracing pre-analysis more narrowly here and are shunted onto the other track, which means that we devote attention to linear order in the number system and, through measurement, nesting of intervals and regions.

We understand that "inequalities" and their "solution sets" are already in the vocabulary of primary school children. The "conjunctive" inequality

$$a < x < b$$

should be included, if it isn't already, and problems worked which lead to names for intervals. With this preparation, a real valued function of a real variable (actually limited to rational numbers) can be seen to create a set function. What, for example, is the image of the interval $[a,b]$ in the function f? "Number lines" and graphing are by now, we assume, familiar, so the notational record keeping can be bolstered by graphs. Graphs, it must be noticed, are often inadequate and it should be emphasized by examples that it is the reasoning that really counts. Consider the image of (0,1) in the function

$$f: \quad f(x) = \frac{1}{x} \quad 0,2$$

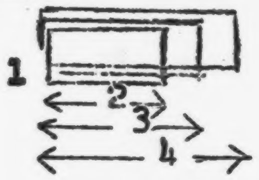
which is not graphable in toto on the usual rectangular coordinate scheme. That image does exist, however, as an open interval and is visualizable on a different kind of graph, say by polar projection of the plane onto a sphere. Children should be brought to realize that it is the mapping which contains the content of such an exercise and that the graphs serve only as visual models helpful for keeping things straight.

Whether explicit calculus problems can be introduced by the sixth grade or not can be learned only by trial in the classroom. We conclude these remarks by suggesting some projects which might be tested - the projects being under examination, not the students. Consider the following simple, indeed somewhat crude, example of an integral operator. Let f be a linear function (real valued on a real interval and nowhere negative), so that the region "under its graph" is a trapezoid. The area of a triangle being easy to compute leads to the constructability of the new "area function".

$$g: \quad g(x) = \int_a^x f \qquad a \leq x \leq b$$

which can be given the notationally reasonable designation $\int_a$. Any such linear function can be thus integrated, and the operator $\int_a$ appears in the role of a function. It would seem that curiosity might arise as to the possibility of integrating a quadratic function, and the difficulty encountered in computing values of $\int_a f$ in this case (assuming the skill of the teacher is adequate to the task) could motivate a more intensive study in later grades of area as a measure of the size of a region.

In order to illustrate this in greater detail, suppose a large collection of rectangles, each member of which has the same altitude, is arranged so as to appear like



This can be interpreted as a picture of any one of several functions,

Rectangle → length of base etc.,

and, in particular, look at the functions

b = Length of base $\xrightarrow{H}$ altitude of rectangle = H(b)

b = Length of base $\xrightarrow{A}$ area of rectangle = A(b)

Partial tables of H and A are,

b	H(b)	b	A(b)
2	1	2	2
3	1	3	3
4	1	4	4

and these functions have, we suppose, been given names heretofore, so what we see is the relation between functions $H \longrightarrow A$. Relations which symbollically look like this have been given names before, so why not call this relation $\int$ right away? Thus

$$H \xrightarrow{\int} A = \int H.$$

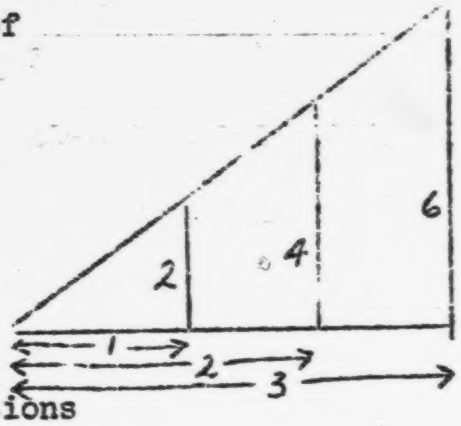
And why the curious ogive shaped symbol? Well, why not? With sufficient "preconditioning" for freedom in the use of symbols in games, it just may seem to be fun.

A sequence of examples which develop the $\int$ relation is fairly direct now. If $\int$ maps the unit function to the identity function, it is reasonable to inquire what it does to the "2" function and other constants. The perplexing problem of notation is hard to decide. Perhaps eventually a tabulation of the form

H	$A = \int H$
I^0	I
$2I^0$	$2I$
$\vdots$	$\vdots$
kI^0	kI

or some adaptation of it could serve to preserve the essential function aspects of the traditional statement $\int_0^x k dx = kx$. Any notation whatever injects irrelevant features into the things it stands for and exacts a price for its practical utility. As it lends clarity to and facility in the use of its objects, it actually does in math classes usurp too much of the primary role of the thing it denotes, and yet this primary role cannot be played without symbollism, very likely cannot be even conceived without it. Unresolvable dilemma!

Continuing along a scale of functions, try triangles next. They take on the appearance of



leading to the two functions

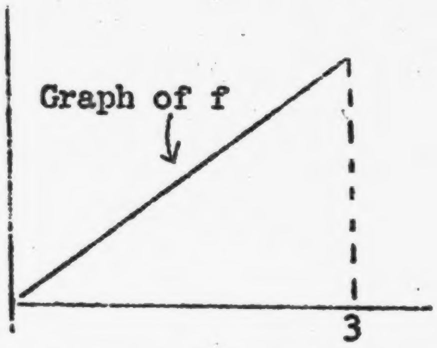
b	H (b)	b	A (b)
1	2	1	1
2	4	2	4
3	6	3	9

where H is the doubling function, $A = 2I$ and A is the squaring function $A = I^2$. Thus $\int 2I = I^2$. Further trials with other sets of right triangles nested in this way would lead to the function relation described by $\int kI = 1/2kI^2$.

It is hard to say at what point, or in what direction, to branch off into other important aspects of integration, be they substantive or notational, but any such next aspect ought to be, while simple enough to be accessible, clearly non-trivial. Extended scrimmage is necessary, but it should never to too long between games.

Suppose the last example is reversed, i.e. given the identity function, what sort of diagram would lead to its image under $\int$? This presupposes simple algebra and graphing, and could go as follows:

Let $f(x) = x$ $\quad 0 \leq x \leq 3$, and picture the function on a graph. Three right triangles can be put in the figure so it looks just like the last one drawn above.



This is nothing new. Suppose other triangles were nested in the diagram as before, but of different sizes, does the image of I under $\int$ come out to be 1/2 I^2 independently of how big the triangles are? At this place the "arbitrary typical" triangle, measured in formula language would yield,

b	H (b)	b	A (b)
x	x	x	$1/2\ x^2$

which, by some mental alchemy, should transfer to

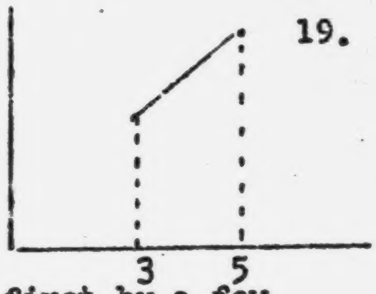
$$I \longrightarrow \int I = 1/2\ I^2 .$$

The hope (pious?) is that this last sentence says more about the nature of the $\int$ relationship between functions than a comparison of the partial tables for I and 1/2 I^2 did.

Remember, we hold no particular brief for the nomenclature in which this is being written at the moment, but only that something has to be put on paper which indicates a route through function and into calculus (end of disclaimer).

After a similar referral to the rectangle case, i.e. starting with the function kI° and reaching $\int kI^{\circ} = kI$ independent of how the rectangles are inserted, we proceed consistently starting with f and aiming toward $\int f$. Here is a place to vary the domain of f and get a version of the definite integral of f.

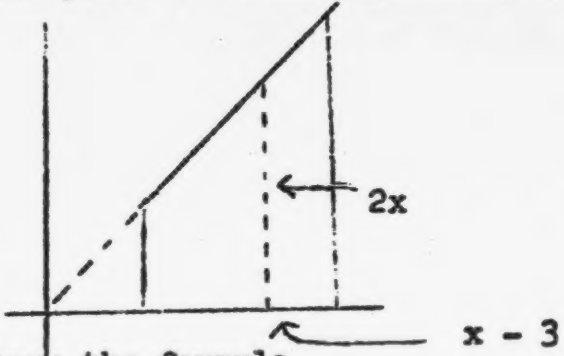
Suppose f: $f(x) = 2x, \; 3 \leq x \leq 5$ with graph:



We compare the height and area functions as before, first by a few cases, (using trapezoids)

x	H (x)	x	A (x)
3	6	3	0
4	8	4	7
5	10	5	16

and it's not very clear whether we have seen A before or not. This may be an object lesson in the power of a more general, symbollic method. Look at the figure



The area function ought to have the formula

$$
\begin{aligned}
s \longrightarrow A(x) &= \text{ave. of altitudes x base} \\
&= 1/2[6 + 2x]\,(x - 3) \\
&= x^2 - 9
\end{aligned}
$$

(or add the rectangle to the top triangle, or however the kids get the area of a trapezoid), which can be recorded in this case as $\int f = I^2 - 9I^0$.

By now there's enough variety in life to require some cataloguing. Suppose several trapezoids like the last one have been treated, all with tops taken from $2I$,

	f		f
$2I$	$[0,5]$	I^2	$[0,5]$
$2I$	$[3,5]$	$I^2 - 9I^\circ$	$[3,5]$
$2I$	$[0,8]$	I^2	$[0,8]$
$2I$	$[3,8]$	$I^2 - 9I^\circ$	$[3,8]$

and some more like this. Apparently the area function corresponding to $2I$, on some domain, in $\int$, has the same domain and changes form depending on the initial point of the domain. It is reasonable to agree on a distinguishing symbol, and write ($0 \leq a < b$) $2I, [a,b] \xrightarrow{\int_a} \int_a 2I = I^2 - a^2 I^\circ$, $[a,b]$.

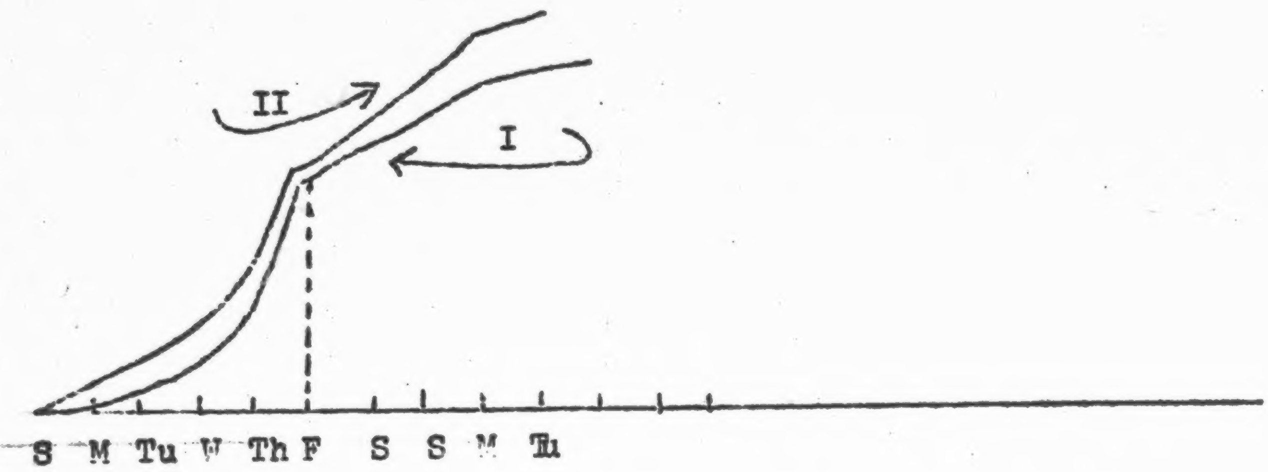
We have tried our hand at the invention of examples leading to the derivative of a function but only with questionable success. However, the idea that is embodied in the words "how fast", some version of it at least, must exist in classroom K. That some things go faster than others is obvious to the very wee children, and so also they see that some things grow faster than others do. Our aim is to get them to use functions, measurement and numbers to quantify the idea, to define it and, ultimately, to relate it to the rest of calculus.

In the spirit of this paper it is presumed that any class of kids is stockpiling functions, and should soon have useful ones lying around for whatever purpose. Exactly which functions would best serve the purpose of pinning down rate at the beginning is impossible to say when thinking in the vacuum we are here. However, it would seem that a class might compare the heights of two (or more) bean stalks recorded day by day to get tables such as

day	height I	height II
1	1/2	0
2	1	1/2
3	2	1 1/2
4	3 1/2	3
5	5 1/2	5 1/2
.		
.		
.		

wherein the first plant started first but grew more slowly than the second. These are global observations. There are others, too, such as the monotanicity induced by the nature of the experiment.

In the context of "how fast" if would probably occur to a class to ask "how much growth each day" and this leads at once to first differences. Thus local behavior of the functions can be observed with a resolution limited to one day. Doubtless this work would go along with the area studies of Section 2 and a graphical record could be kept. It will help to move the project along if weekends are included so a couple of days are skipped now and then. Graphs could be



with points joined by line segments. In order to get the daily growth over the weekend, a difference quotient would necessarily arise, so the daily rate measures would be recorded as

day	Δday	I	ΔI	$\frac{\Delta \text{I}}{\Delta \text{day}}$	II	ΔII	$\frac{\Delta \text{II}}{\Delta \text{day}}$
1		1/4			0		
	1		3/4	3/4		1/2	1/2
2		1			1/2		
	1		1	1		1	1
3		2			1 1/2		
	1		1 1/2	1 1/2		1 1/2	1 1/2
4		3 1/2			3		
	1		2	2		2 1/2	2 1/2
5		5 1/2			5 1/2		
	3		4	1 1/3		5	1 2/3
8		9 1/2			10 1/2		
	1		3/4	3/4		1	1
9		10 1/4			11 1/2		
.							
.							
.							

The points to notice being that the first differences measure how fast the plants are growing and that these are useful as long as the time intervals are the same, but that over the weekend the time interval is different and to get a useful consistent measure of "how fast" the difference quotient is the better. On the growth graph, this ratio appears as the slope of the line segment over the corresponding time interval.

More generally, one might have the students sketch a tangent line on the graph of a smooth function, measure its slope and thus construct a rough approximation to the derivative function, and with this background illustrate what the derivative operator might be. In a context of mechanics this activity can be viewed, quite properly, as a graphical study of rate problems and is all to the good. However, in order to proceed closer to the "pre-analysis" line, attention should be given along with this, if not before it, to "inverse image of intervals" games. These games should ultimately contain the question, "Given f and the interval I in its range, what are some intervals in the domain of f whose images are contained in I?" It is epsilontics readiness that we are creating, and there is no escape from the necessity for it. This technique, and the concept it treats, in whatever form it appears, is the distinguishing characteristic of analysis and must be faced sooner or later. The sooner the better.

By sixth grade what else should, and could, be done to prepare the children for calculus? Sequences have not been mentioned here, and perhaps they ought to be. Experience with college students underlines the need for preservation of the greater openmindedness they seem to have had as children. Presently students seem to have been conditioned to get an answer to a given math problem by a well-learned procedure, in a small finite number of steps by a small number of deductions. True, it's easy to find the tangent line to a polynomial curve by the naive process - just write a new polynomial (derivative), evaluate it to get a number (slope) and substitute into the point slope line formula. But this avoids calculus. To generate curiosity about the existence of a tangent line, or more basically its very definition, and to generate a notational technique that facilitates such a definition more than abfuscating it is a matter of a different order.

So perhaps, on a much lower spiral arm, it may loosen things up to start with some practice with sequential problems, such as:

Find a point on the number line (not 0) closer to 0 than 1 is. Find lots of these and list them. How close to 0 is each of these answers? Which is the closest one of those found? Find one closer than that, etc. As soon as possible this must be done more systematically, by use of formulas, but to get less trivial cases, such exercises should be extended to other points of accumulation than zero.

Introduce a measure of "closeness", in the example above, say, 1/5; then find several numbers closer to 0 than that and eventually agree that if such a measure is e, then e/2 is still closer, and that e/2 is one such number for each e.

Try nesting intervals; take 2 and let [1,3] be an interval containing 2. Find several more intervals containing 2 and compare them (overlapping, nested, disjoint?). Look for a shortest one, and then get one shorter than that. Given e, find intervals shorter than e, all containing 2. Play the "e game" - given some e, find a point closer in than e, or an interval shorter than e. Ease with these games should increase with familiarity.

Given the sequence $\frac{n-1}{n}$, it should be discovered that it is increasing, bounded, and that given a measure e, the sequence eventually stays closer than e to 1. The success of this process will depend on the ingenuity with which sequences and nests can be concocted which are simple enough and yet possess variety enough to be intriguing.

Before this palls and after the essentials of the test for a limit of a sequence are familiar, sequences on two number lines can be compared where the number lines are connected by a mapping. Suppose the lines are X and Y with the mapping $y = 3x$. For the sequence $x_n = \frac{n-1}{n}$, what is the image sequence for y_n ? Lots of these. Compare the "e games" on the two number lines together. The nested interval game near $x = 0$ for $y = \frac{1}{x}$, and for $y = \frac{x}{x}$ near zero should be interesting variations.

The object of this is to build into childrens' conception of what goes on in math some understanding of limit techniques as a bona fide part of the business along with the many other activities they come to feel are really math. This can be done with geometry, too. For example, how small should squares be if when they are packed into a right triangle their total area comes short of the area of the triangle by less than e? For curved figures this e question could take the form, how small must these squares be if the total area of the covering minus the inscribed squares is to be less than e? Another project could be to find and test by the e game a set of nested intervals condensing on 2.

These problems should all be chosen so the e game can be played; otherwise they are little more than drill on arithmetic fundamentals, and the aim here is to create, as we remarked earlier, an epsilontic readiness.

APPLICATIONS

Mathematics presents an intellectual challenge that is stimulating to many, even to youngsters. This, by itself, makes the subject interesting and viable in elementary school even if it should seem to be void of practical applications. However, the number of young students intrigued by mathematics, and the level of interest of all, is increased greatly if the importance of its applications is made evident from the earliest grades. The arithmetic skills may be better learned while doing the very things for which they are important, instead of through sterile drill. It is then important to do the calculations speedily and accurately to get at the interesting results ahead. Also, abstract questions of sets and algebraic structure attain increased importance if they lead to some technique and power that permit one to predict the probability of an event or the motion of a particle.

Geometry, especially at the elementary school level, is closely associated with its application to the approximately Euclidian space in which we live. The ability to describe, construct the objects that surround us, and to predict the results of their composition and decomposition, is an application of some immediate interest to children.

Functions and analysis are the tools of most of the applications in science and technology. The concept is introduced to youngsters in terms of the physical objects for which the functions are models - children and classrooms, areas of figures, rolling wheels, particle trajectories. Measurement is an application of geometry and functions. It also involves probability, itself an application we will discuss below.

In the presentation advocated in this report, all of the material involves application. However, it is also our purpose to describe a part of the curriculum in which the topic is dominated by the application, which may need various types of mathematics for its exploration. The two subjects of this type discussed in some detail are probability and mechanics. These two differ from each other considerably in the type of mathematics used, and also in the proportion of mathematical concepts to physical or scientific concepts involved.

Probability is in one sense a mathematical discipline, which can be axiomatized and treated in the abstract. But when its principles are evolved experimentally it is a problem in modelling - a central aspect of applications. Furthermore, analyzing distributions of events and answering questions about expectations requires the application of the arithmetic operations, averaging, graphing, functions, real variable, set operations, permutations and combinations, symmetry, and other mathematics. The physical experiments to be done are simple, require only a little technique, and can be associated immediately with some mathematical manipulations. Some care has to be taken as to whether events are independent, generated in a consistent manner, and unbiased by external factors. It must be established that the physical events correspond approximately to the assumptions of the model,so that the experimental, physical aspect must not be slighted. But this aspect is not as difficult as the specifically mathematical problems involved in the probability unit described in this report.

The science of mechanics demands a great deal more of physical experiments in the classroom. In modern times there are few people who are interested in mechanics as a deductive discipline, although in the nineteenth century it was treated, at times, as an extension of geometry. To abstract the principles of kinematics and dynamics a long sequence of experiments is required. Many of these involve painstaking measurement, but only trivial mathematics. Friction, rolling constraints, the lack of weightless ropes, etc., introduce many complications into experiments that would otherwise lead more directly to simple mathematical formulation. The temptation must be avoided of drawing conclusions which the experiments actually performed do not indicate. Instead one must do more difficult experiments, get along with fewer results, or use guesses as working hypotheses. In this last case, the experiments performed up to the time of making the hypotheses only indicate the hypotheses as possible extrapolations from the data. This is only worthwhile if one can eventually predict from the model, perhaps after considerable mathematical reasoning, events which can be verified by experiment in the classroom. While there are many loopholes in guessing at Newton's laws of motion from spring and inclined plane experiments, the predicted parabolic motion of a particle in a uniform force field can be closely verified by the motion of a solid object through air over the span of a classroom. There are no rolling constraints and friction is small. But the connection of the parabolic orbit to the laws requires much more mathematics than does straight line motion and must come after several years of preparation.

Other topics may present very fruitful applications of mathematics for elementary school. Some of these appear briefly in some of the other units discussed in this report. The discussion of functions uses daily plant growth as an early example, and later applies similarity and trigonometry to measurement. There will be many opportunities in the course of all the mathematics described in this report to make brief applications.

It is highly desirable to do so providing that it sufficiently illustrates the power of the mathematics at hand and that it does not distort the meaning and use of the application. We have concentrated on two problems to provide a comparatively full description of the richness of the mathematics evolved in several years of development of an applied topic; and also to show that no violence need be done to the non-mathematical aspects of the topic.

The relative timing of the experience with various mathematical and physical ideas is very important in this area. The child's movements and playing with blocks and other toys are his earliest introduction to geometrical and physical concepts and develop the intuitive response to these phenomena. This must come before any more explicit approach to the mathematical or scientific content. On the other hand, a certain facility with numbers is required before measurement can be used to quantify any of the scientific conjectures. This interplay is evident throughout the grade levels in the applications.

The importance of permitting time for an experimental and discovery approach may well be more important in the applications than it already is elsewhere in mathematical education. The process of modelling is delicate and all but unteachable in any direct way. An intuition for modelling can be built by allowing time for trial and error and for multiple approaches that stimulate inductive reasoning. It is doubtful if the classical method of teaching by example (case histories of modelling) is adequate. Direct experience of the struggle seems to be a requirement.

As a consequence in the probability unit, it is appropriate to let the child who wants to, to experiment with different objects in the cups, differently marked, without replacement, etc. In short, with any set of rules he may think of, even if the teacher is aware that that experiment is a "bad" one - in the sense that it may lead to complex, confusing, or biased results. This is part of the problems of experiment and modelling that a student needs to find out for himself. Similarly in the early mechanics many blocks, springs and balance boards should be available to allow the testing of some hypotheses, e.g. the effect of the color of the balance arm on weighing, that may occur to the student. The connection may seem unlikely to the teacher, but many important connections have seemed unlikely in earlier times. What connection did light have with lodestone to the pre-Maxwellian intellect? The mental freedom to cast about for associations is essential to research.

The mathematical models invented by children in the classroom also must be treated with respect and the readiness to understand what experience they are trying to express. The feeling, expressed by ten-year-olds, that bodies reach a maximum velocity under a constant force is a deduction from their experience in pushing things and trying to push as hard while the speed increases. Care must be taken not to "prove" to them that they are wrong in any absolute way. After all, terminal velocities are reached in a frictional medium when the applied force is steady. Also relativistic effects do lead to a common maximum velocity, although this effect will not be seen in the elementary classroom.

In summary, the classroom technique, as well as the curriculum material, must indicate not only the variety of mathematical and scientific content to be explored eventually, but also the variety of ways in which individuals examine evidence, cogitate, hypothesize and generalize. If some of these basic features of the researcher can be developed in the student, mathematics will have been well served in two ways; by the awareness of the importance of the subject, and by the growth of an ability to reason fruitfully as well as accurately. The latter is equally important to the scientific disciplines which have been invoked as an application of mathematics.

Section II - Chapter 4

PROBABILITY

Today, probability is one of the most widely used branches of mathematics, not only in various vocations, but in the everyday life of "the Man in the Street." The ordinary citizen is constantly bombarded with statistics about toothpaste, automobile accidents, the probability that there is a connection between smoking and various kinds of illness, the probability that candidate A is going to win an election, etc.

As well as being useful in the real world, probability is an interesting and exciting means of getting children to practice arithmetic. It is also a good mathematical model of the real world, and offers children considerable practice in creating mathematical models with approximate reality.

All of these reasons seem to point to the early teaching of some probabilistic concepts in the elementary grades. Certainly, a considerable amount of probability should be learned by all pupils before some discontinue their formal mathematical education. A further reason for the early introduction of probability into the curriculum is that many people have the feeling that mathematics studies only exact data and exact numbers -- probability will give the feeling of studying distributions and uncertainties before the pupils become overly enamoured with "getting the exact answer."

It is our belief that the study of probability (as well as the early study of other mathematics) ought to be closely associated with the real world. This means that the children will perform many experiments, and will attempt to draw mathematical conclusions from those experiments. In the early grades, the mathematics will be of a very informal nature, and the children will be getting a feeling for certain concepts, without necessarily stating them explicitly. At a later time, more explicit, quantitative conclusions will be drawn, and analyzed.

The following sequence of events might be appropriate for grades 3 through 6. A large amount of deduction is required by the 5th and 6th grade materials. Experience may show that, even for properly prepared children, this part of the unit is more successful a grade or even two grades later.

GRADE 3

The purpose of the early experiments in the third grade would be to develop a feeling for the long range stability in a situation in which each individual event is unpredictable. For this purpose, a variation of the thumbtack throwing experiment would be used. (See Estabrook Progress Reports. A forthcoming C.C.S.M. Volume will give details of classroom presentation.) The advantages of the thumbtack are that the children do not have a preconceived notion of what the probability OUGHT to be, and yet the long ratio of successes to trials will become quite stable (if care is taken to use the same method of throwing the tack each time). Another advantage is that children can get some feeling for the connection between the physical situation and the results by varying the length of the tack (using a coin as the limiting case in one direction, and a finishing nail as the limiting case in the other direction). For young children, the obvious danger in using thumbtacks may outweigh the advantages, and other objects can be used for the same purpose. For example, small corks in which the circular bases are relatively large with respect to the height can be used, and other corks with different shapes can be used to get the same effect as varying the length of the thumbtack.

The arithmetic involved in this experiment is little more than counting. However, a strong feeling for ratio should be built up as the children note that (e.g.) 3 out of 10 times, the point is up; 42 out of 100 times it's up; 391 out of 1000 times it's up, and so on.

After the relative stability of the long run ratio has been accepted by the children, it will be worthwhile to see if they are inclined to make predictions for specific events. For example, if they believe that in the long run about four-tenths of the trials will come out with the point up, and it happens that the last six events have all resulted in the point's being down, do the children think that the probability of getting a point up is still .4, or do they think that it is greater than . 4 (or, perhaps, less than .4). If a careful enough, and long enough, record is kept, it should be possible to see that, in the long run, after the tack has landed with the point down six times, it will land with the point up about .4 of the times on the seventh trial (if that is its probability generally). Thus, the children should get a feeling that the "law of probability" has to do with the long term ratio, not individual events.

Other experiments, of a similar nature which can be carried on in the third grade include placing a number (say 10) colored corks in a malted milk cup, drawing one without looking, recording the results, replacing the cork, and starting over. Experiments of this type have been tried in Ithaca, New York schools, by David Block. Again, the long range ratio of red corks to the number of draws should become rather stable, and in this case, the ratio should stabilize around the expected number,(the true ratio of red corks to corks in the cup). If more than two colors of corks are used, the data may be difficult for the children to accumulate -- this can be simplified by providing the appropriate number of pegs, colored with the same colors as the corks and placing a washer over the appropriate peg whenever that color cork is drawn.

Games can be played using the cork drawing experiment, in which the teacher places corks in the container and the children try to guess, after a few drawings, what the proportion of various colors of corks is. With only ten corks and two colors, the children will find it advantageous to guess rather wildly. Therefore, it will usually be desirable to use at least three colors. Other games can also be made up rather easily. For example, each child can have his own cup, make up his own distribution and then let another child try to guess the distribution. Rules for these games are quite flexible -- each child can be allowed one guess -- if it's right he wins, if it's wrong, he loses; each child can guess after each drawing with the first right answer winning, etc.

Other variations on the cork drawing experiment are also possible. For example, one could number the corks and then determine the probability of getting corks in various subsets, and intersections and unions of various subsets. That is, the pupil might calculate the probability of getting a cork which is red and has an even number associated with it; a cork which is either green or has a number divisible by 3 associated with it, etc.

In general, the purpose of the experiments in the third grade is for the children to acquire a feeling for the long term stability of the ratio of successes to trials in probabilistic events. They should also realize that the ratio can often be predicted closely by using a prior argument of symmetry and intuitive compounding. Many other experiments involving dice, coins, cards, etc. are possible if time permits. The children should learn to keep careful records, and should learn to interpret those records with some degree of sophistication. At the beginning, the teacher would keep the records and help with the interpretations, but these activities should be turned over to the children as soon as possible.

GRADE 4

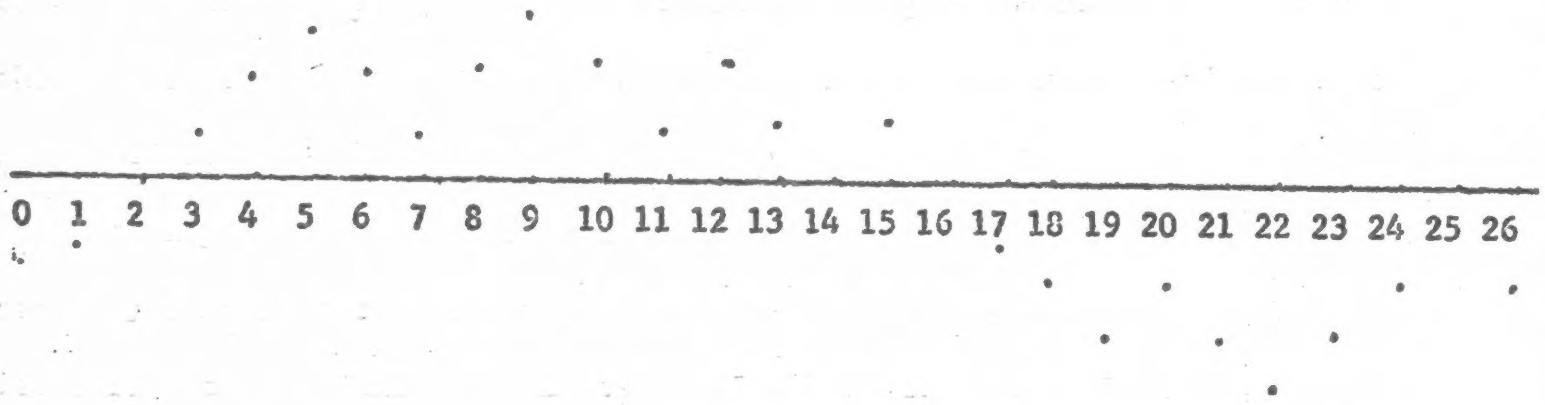
Measures of dispersion, randomness, and sampling procedures will be the main concepts developed in the fourth grade probability study.

The range is the most obvious measure of dispersion for a set of data, but after having suggested this, and used it to measure the dispersion of several sets of data, the children will probably feel that some better measure could be found. The data on which the measures are tried can be collected from more experiments such as those used in the third grade, or by the collection of data from other sources (e.g.,have all the children measure the length of a table and record the answers independently, record the heights, weights, or ages of the children in the class, have each child flip a coin ten times, and record the total number of heads).

As they plot the data from their experiments on a graph, the children may notice that one or two pieces of data tend to increase the range greatly, and don't seem to be significant in terms of all of the data. That is, two distributions may look very similar except that in one, one or two data may be much further from the mean than in the other distribution. In spite of this, the children may have the feeling that the dispersion in the two distributions is essentially the same. This will be more apparent to them the larger the number of data in the distribution. From these facts, two other measures of dispersion can probably be elicited from the pupils. First, a "trimmed range," in which the two data which are furthest from the mean at either end of the distribution are simply removed before the range is determined. The second measure of dispersion would be quartiles.

Once a measure for dispersion has been decided upon (or several measures have been considered), it will be interesting to see how the dispersion is affected by increasing the number of data. For example, suppose each child in a class of 25 throws a coin 20 times and records the number of heads, what will be the relation between the trimmed range (or other measure of dispersion) of this data and the trimmed range of 100 such data? Would the trimmed range be the same? Would it be four times as great? Would it be somewhere between these possibilities? Where? Through experimentations, the children can get a pretty good estimate of where it will be.

Another interesting experiment which can be tried at this level is to have a child throw a coin, and record his results on graph paper, adding one to the ordinate of the previous point if the coin lands head, and subtracting one if the coin lands tails. The distribution would look something like this:



If this procedure is continued, relatively few of the points will actually be on the line, but they will tend to cluster around it -- generally fitting inside of a parabola which has the x-axis as its axis of symmetry. This again, should give them a feeling for the non-predictability of individual events, but the long range stability of the proportion of heads to trials. As a variation on this procedure, a thumbtack, or other object, (with probability not equal to 1/2) can be thrown and the slope of the axis of symmetry of the parabola will approximate the probability in question.

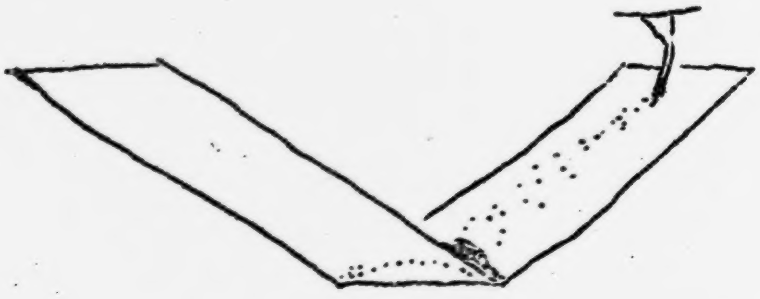
The above experiments lead naturally into a discussion of random walk experiments, and if the appropriate science has been studied, this can lead on into a discussion of Brownian motion, and molecular activity. If this effect could be observed through a microscope (the projecting type would be good with young children) it would be desirable.

Next, a discussion of random sampling would be appropriate. "How would you choose a random sample of people in the fourth grade of this school?" Have each child pick a random sample of ten people, and then collect some data about those people (maybe height, weight, sex, etc.). Then, have them create a random number table, and use the random number table to choose a sample of ten more people.

Collect the same data, and compare the results. Then, compare the averages in the samples with the averages in the total population. Hopefully, this will demonstrate the power of random sampling.

There are many ways to create the random number table. One method is to go through a telephone directory, and choose the fourth digit (say) of the top telephone number in each column. Variations on this can be used, but it is important to avoid possible non-random effects which would result from using a digit in the exchange (one of the first three digits), or by taking names that follow each other immediately since members of the same family may be listed, or large corporations or agencies may have their numbers listed several times. It is also possible to construct a table using a purely constructive means (such as rolling a die and flipping a coin -- H,1 results in 1; H,2 in 2; H,5 in 5; T, 1 in 6; 2 in 7, etc; with H,6 and T,6 being ignored), but it is important to know that the objects are "honest." This means that the proportion of 1's, 2's, etc., should be approximately the same. Of course, it is also possible for the pupils to go to a previously constructed random number table but this is not as instructive as creating their own. Once having constructed the table, the pupils should compare the total number of times each digit turned up then the totals for the first half of the table, and soon. They should also look for runs of two of the same digit, runs of three, etc. What is the average number of digits passed over to arrive at a specified digit from an arbitrary starting point? After a run of that digit?

An experiment in which a normal curve is created is very effective in showing children both the dispersion and the central tendency of random events. One beautiful experiment of this sort is to pour salt (or sugar, or sand, or any substance with small grains that will tend to bounce -- not slide or roll -- down the paper) onto a folded sheet of paper so that the grains will bounce down the paper towards the fold. It is important to always pour in the same spot, and to have the spot quite a distance from the fold, so that a large number of bounces will occur before the grains stop. Of course, the line of the fold should be horizontal. A normal curve will appear on both halves of the paper, and careful observation will show an inverted normal curve in the "hollow" between the other two curves. This experiment has to be tried to be fully appreciated.



One further experiment which can be tried at this time has to do with the way in which people estimate the number of fish in a pond. The procedure is to catch a large number of fish, band them, throw them back, come back in a few days and catch some more fish. From the proportion of fish caught the second time that are banded, a good estimate of the total number of fish can be derived. This same procedure can be used (without as many doubtfulassumptions regarding the psychology of fish) with corks. Given a large number of corks in a container, how can we decide how many (approximately) there are? The procedure would be essentially the same as in the fish experiment except that the corks removed on the first drawing would be replaced by corks of a different color (also different from any which might be in the container, of course). The container would be thoroughly shaken up before redrawing. The arithmetic involved would be a simple proportion, and should cause no great difficulty.

GRADE 5

In Grade 5, compound events would be considered in some detail. The procedure, in general, would be for the pupils to begin with an experiment from which they could deduce an hypothesis regarding the probabilities in a combination of

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events in which the simple probabilities are already known. Having decided upon an empirical hypothesis, the pupils should then be encouraged to consider the problem in a more theoretical or mathematical context, and try to derive a general theory that fits the empirical results very closely. From this mathematical discussion and resulting hypothesis, the pupils should be encouraged to predict results for other experiments which are quite similar, and then predict results of experiments which are quite dissimilar. Then, they should carry out these new experiments and see if the empirical results are close to the predicted results. If not, perhaps a reconsideration of the hypothesis would be in order.

In getting a general hypothesis, it is important that the teacher not insist upon a correct and careful verbalization of the principle, but rather, a good strong intuitive feeling for what the principle is. If the children can predict with some accuracy the results of other experiments, they presumably have a good understanding of the principle involved. On the other hand, if over-emphasis is placed on the verbalization, the children will tend to memorize the words without necessarily understanding the meaning.

The procedure discussed above could be applied to conditioned events in the following way:

Ten corks are placed in a container. The corks are numbered from 1 to 10. Numbers 1, 2, 4, 5, 7, and 9 are green, and the others are red. What is the probability that a cork drawn at random is green? What is the probability that it is red? What is the probability that it has an even number? If you draw a cork that is red, what is the probability that it has an even number -- experiment with at least 20 drawings to get an empirical probability. Draw pictures of the sets involved to clarify the notions. Suppose cork number 9 were red instead of green, how would this change the probability that a red cork is even? Given the corks in the original experiment (9 still green), what is the probability that a green cork drawn at random is associated with a number that is divisible by 3?

Many other experiments might be tried to see if children could predict the outcomes in advance. For example, two chips are placed in a cup. One chip is red on both sides, the other is red on one side and green on the other. If a chip is drawn at random and placed on the table, and it happens that a red face is up, what is the probability that the side which cannot be seen is green? Add three more chips that are green on both sides, how does this affect the probability? Add a chip that is red on both sides, how does this affect the probability? For each of these experiments, the children should first make a prediction, and then experiment to see how accurate their prediction is for a large number of trials.

The case of disjunction should probably be done first with disjoint events, though it may be interesting to start first with events which are not disjoint and then consider disjoint events as a special case. An advantage of trying disjoint events first is that then the children are likely to make incorrect hypotheses regarding the non-disjoint cases, and will have an opportunity to correct a mistaken hypothesis in light of experimental evidence. For this, they might start by considering such disjoint events as getting a two or a three on a single die. The probability of getting either a red cork or a blue cork when there are a known mumber of red, green and blue corks in the container, etc. After they have become quite good at predicting the probability of the disjunction of disjoint events (this should not take long), try a problem such as: There are ten corks in the container, seven are red and three are green. The red ones are numbered from 1 to 7, and the green ones are numbered from 8 to 10. What is the probability that a cork drawn at random is either green or has an even number? Many of the children will answer 3/10 + 5/10 = 8/10. This answer is sufficiently different from the correct one so that a reasonable amount of experimental evidence should lead the children to suspect that something is wrong. Of course, the original experiment ought to be so designed that the obvious mistake will not result in a probability greater than 1, or the need for experimentation will be by-passed.

Conjection of events should be studied first with independent events, such as two dice, preferably colored differently (or thrown separately). Questions might include: "What is the probability that the green die lands with an even number showing and the red die shows a number divisible by 3?" "What is the probability that the red die shows a number less than 5 and the green die shows a number greater than 2?" Etc. Similar experiments can be carried on using other objects. For example, "What is the probability that both thumbtacks land with the point up?" What is the probability that the red tack lands with the point up and the green one lands with the point down (two clors of tacks are desirable for this)?" "What is the probability of getting a green cork from container I and a red cork from container II (with known distributions)?" Etc. If thumbtacks are used, it is important that they be thrown in essentially the same way they were thrown to calculate the simple probabilities.

After the pupils get quite good at predicting probabilities for the conjunction of independent events, they should try some in which the events are not independent. For example, place two red, and one green cork into a container. Shake well. Draw one cork. Leave it out of the container and draw another cork. What is the probability that the first cork is red and the second cork is green? If the pupils believe the probability is 2/9, it may be necessary to perform close to 90 trials to convince them empirically that this is probably incorrect, however, with the entire class working on the experiments, considerably more trials than this can be run in a very short time.

Other examples of non-independent events can be constructed easily. One nice example of this involves gluing two dice together and rolling them -- the dependence is quite clear in this case. Other examples include: Throw a die, what is the probability that the number showing is both even and greater than three? Tie the points of two thumbtacks together with a very fine, relatively short thread and determine the probability that both land with the points up. Make the thread longer and repeat the experiment.

In the theoretical discussion of the various kinds of experiments discussed above, children will find it very helpful if pictures of the sets and subsets involved can be drawn on the board to make it clearer why the particular results they are getting seem reasonable. This is particularly true in the case of conditional probabilities and conjunctions of non-independent events, but will be valuable throughout the discussion. As set union and intersection are useful concepts in this context, this is one of the more fruitful early applications of the set theory now being taught in elementary school.

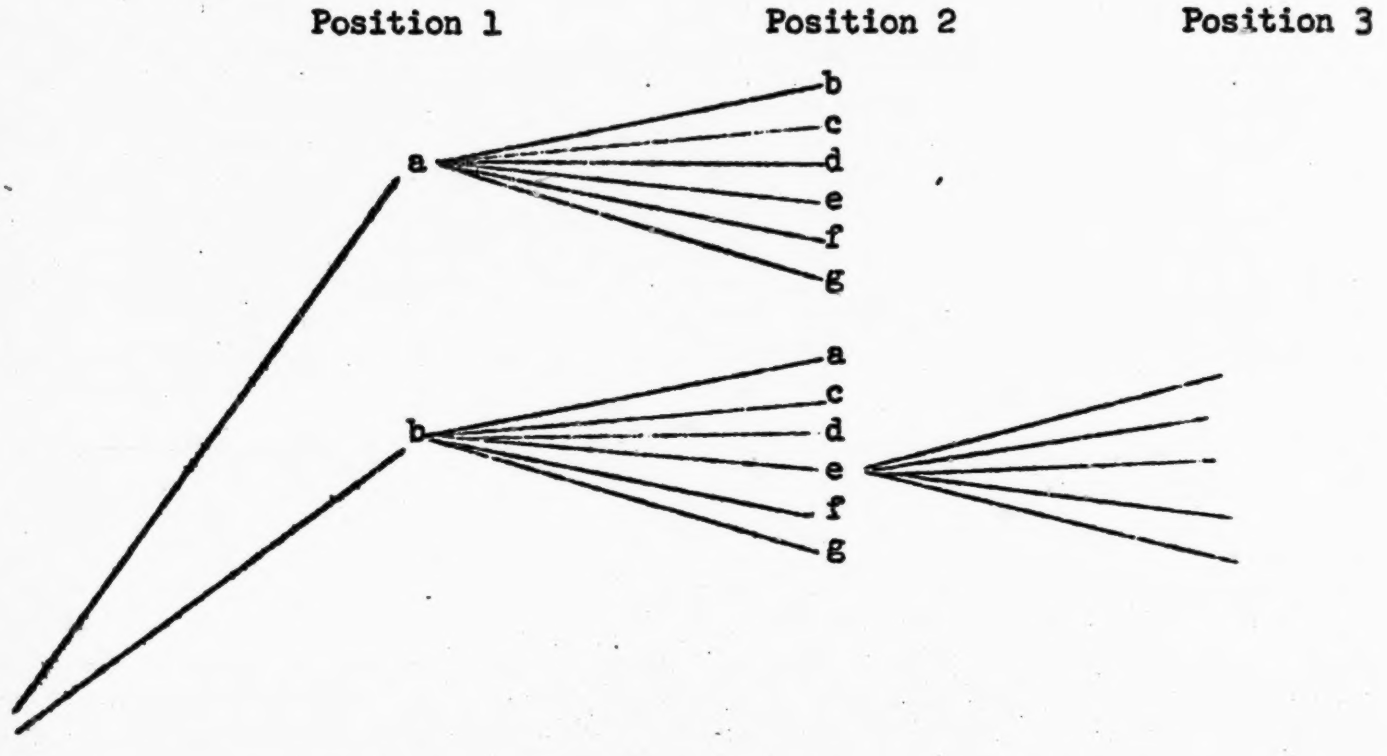
GRADE 6

As stated in the introductory section of this paper, the deductive nature of much of the material suggested here for the 5th and 6th grades makes its success in these grade levels even in the "Goals" context, difficult to forecast. The experiment seems worthwhile to us because of the strong backing of experience and intuition to the reasoning required in this work.

If the principles developed in the fifth grade regarding compound events were not stated explicitly at that time, they should be redeveloped quickly (without as much experimentation being needed) and the children should be encouraged to state them explicitly so as to make the job of applying them to more complex situations easier. Then, these principles can be applied to such problems as "What is the probability of throwing a seven with two dice?" "What is the probability of throwing a five with three dice?" "What is the probability that if five coins are thrown, exactly three of them will land heads?" "What is the probability of getting a thirteen with two 12-sided dice?" For this last experiment, it will be necessary to discuss the question of what is a 12-sided die. Some of the children will have calendars at home which are printed on regular dodecahedrons, so this discussion should not be difficult -- of course, the teacher can acquire one of these in advance if he wants to. Then, the discussion can turn to the question of exactly how many regular solids exist. The usual procedure of demonstrating that there can be at most five (three with triangular faces, one with square faces, and one with pentagonal faces) can be discovered by the children quite easily if they are asked to consider the number of degrees in the angles meeting at a vertex. Children can figure out the patterns for constructing the regular solids, and can do a good job of constructing them. Before trying experiments involving compound events, they should test the simple probabilities to see whether the probabilities are approximately what they should be.

Now, ask the children to determine the probability of getting a twelve with five 4-sided dice. The problem is difficult enough so that very few, if any, will be able to succeed, and yet, all will know the general principle involved. Then, try to analyze the problem with them. Set down the favorable cases in some sort of rational order. For example, start with the largest possible number, and make the number get smaller, or remain constant as you move from left to right (monotonic): 4, 4, 2, 1, 1; 4, 3, 3, 1, 1; 4, 3, 2, 2, 1; 4, 2, 2, 2, 2; 3, 3, 3, 2, 1. This part is relatively easy. Now, how many ways are there of rearranging the first set of numbers? They can start by simply trying to find all the rearrangements, but this will be relatively unsatisfactory. From this, it should be clear that some method of studying rearrangements from a mathematical point of view would be desirable.

Once the need for a process of determining the number of rearrangements of a set of objects has been established, we go back and consider simpler problems. First, how many ways are there to arrange four objects in a row? Try it. The children can either use real objects, or better yet, use symbols on paper -- thus making it easy to keep track of which arrangements have been tried, and how many have been found. After the children have established the fact that the number of rearrangements of the four objects is 24, suggest that perhaps they can find a pattern which would have told them this without all of the work of actually putting down all 24 arrangements. There will probably be several such patterns suggested, all correct (and maybe some incorrect ones -- all should be checked to see that they give the correct answer, not only for four objects, but for three, two and one objects). Correct patterns should not be discouraged, even though the teacher feels that they will not be fruitful in the long run -- let the children find that out for themselves. Next, try the number of rearrangements of seven objects. Is there some way of analyzing this situation? Probably the most fruitful method, which the pupils may discover with some encouragement, is a tree diagram. Suppose the objects are labeled a,b,c,d,e,f,g. There are seven positions, each to be filled by one of the objects. How many choices are there for an object to fill position 1? Suppose a is placed in position 1, how many objects are left with which to fill position 2? Suppose b had been used to fill position 1, how many objects would be left with which to fill position 2? Suppose c had been used in position 1, how many objects would be left from which to choose for position 2? This information can be summarized as in the following diagram:



If the diagram is drawn in some detail, it is clear that since there are seven choices for the first position, and for each of those seven choices there are six possible choices for the second position, there must be 7X6 or 42 ways of filling the first two positions. Young children may think of this as 6 plus 6 plus 6 plus 6 plus 6 plus 6 plus 6 at first, but that is perfectly all right. Now, continuing the process, suppose b is chosen for the first position and e for the second position, how many ways would there be to fill the third position? Would this be true for each of the 42 ways of filling the first two positions?

Continuing this procedure, the children should see that the number of different arrangements of n objects is n!. At this time, it is probably not desirable to introduce the usual notation, but rather wait until the pupils have written out 10x9x8x7x6x5x4x3x2x1, etc., several times and see some need for a shorter notation. Oftentimes, a short notation (such as this) can obscure relatively simple concepts in the minds of young children.

Now consider the question of how many ways there are to arrange six objects, three of which are identical to each other. Suppose the objects are a,a,a,b,c,d. If the three a's were labeled so that they were distinguishable, how many ways would there be to arrange the objects? Of those 720 ways, how many correspond to the one arrangement a,a,a,b,c,d that we would like to count when the labels are removed? That is, if the a's were distinguishable, how many arrangements of a's would correspond to this one arrangement where the a's are not distinquishable? Would this be true for any other single arrangement we'd like to count (such as a,b,c,a,a,d)? Then, in counting the 720 arrangements, how many times as many arrangements did we count as we wanted to count? Then, is the answer we would like just 1/6 of 720? Using the same procedure, analyze the problem for a,a,a,b,b,b. Then, let the pupils consider such problems as a,a,a,a,b,b; a,a,a,b,b,b,b; a,a,c,c,c,b,b,b,b,d; etc., with explanations of each step in their reasoning.

It is entirely possible for them to answer all of the questions in the previous discussion without really understanding what they are doing, however, a complete discussion, by them, of similar cases should help to clarify the idea. Some will still wonder why such a roundabout way of attacking the problem is used, and this is good -- encourage them to try to find a more efficient method.

There are many problems available in standard textbooks for pupils to use in practicing their knowledge of permutations. For several of these problems, they should actually write down all possible arrangements to show that experience corresponds with the theory. Then, they can go back to the dice problems, and decide probabilities such as getting a 15 with five 4-sided dice; etc.

After this section is finished, have the children expand each of the following (using the distributive law): $(a+b)$, $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, $(a+b)^5$, $(a+b)^6$. Now, look for a pattern. With a small amount of encouragement they should be able to come up with Pascal's triangle, or something equivalent. Then, reconsider one of the expansions without using Pascal's triangle ("suppose you don't know the expansion for $(a+b)^4$"). It should be clear that $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b) = Pa^5 + Qa^4b + Ra^3b^2 + Sa^2b^3 + Tab^4 + Ub^5$ where P, Q, R, S, T, and U are numbers. The remaining problem is to determine what those numbers are. In order to determine R, for example, we would note that the three factors of a can come from any three of the original factors, and the two factors of b must come from the other two original binomial factors. Thus, if the factors are written in the order of their "parent" factors, the possible ways of getting a^3b^2 are aaabb, aabab, abaab, baaab, aabba, ababa, baaba, abbaa, babaa, and bbaaa; or, simply all of the ways of arranging the letters aaabb. In each of the other cases, the reasoning would be similar, and the children should, through similar reasoning, be able to convince themselves that the binomial theorem is true.

After some work with the binomial theorem, the pupils can again be asked to compute the probabilities for all possible events when five coins are flipped. Then, the same problem can be tried with thumbtacks (with known probabilities not equal to 1/2 for individual events). Presumably, a relationship between this problem and the binomial theorem will be noticed, and should be discussed quite explicitly. Binomial distributions can then be computed for various values of p, and N (the number of trials). The pupils should be able to determine what conditions have to be met in order to have a binomial experiment -- e.g., independent trials with p remaining constant, a predetermined number n, etc.

Now, they are ready to test various hypotheses. For example, consider the question of spinning a new penny on a flat surface. Most people would assume the probability of the penny's landing tails is 1/2. Suppose, however, that somebody claims that the probability that the penny lands tails is really .8. How would you test to see which hypothesis is true? Suppose the true probability is 1/2, if the coin is spun 10 times, what is the probability that the coin will land tails either 10 times, 9 times, or eight times (i.e., eight or more times)? Then, if the coin lands tails eight or more times, can you be about 95% sure that the true probability of getting a tail on any one spin is greater than 1/2? That is, is the probability about .05 that the true probability is 1/2? (Incidentally, for a relatively new United States one cent piece, the probability of getting a tails with a spin of the sort described here is considerably greater than 1/2 -- depending on how worn the edge is, it can approximate 1. For Canadian Pennies, a reverse minting process is apparently used, and the results are reversed.)

Further experiments can be tried of a similar nature, and binomial probability tables can be created for various values of N and p.

Simple experiments can be designed by the children to test to see whether somebody has extra-sensory-perception, to predict with a certain degree of certainty that there are a given proportion of red corks in a container, etc. The question of being fair to two contradictory hypotheses (say .5 and .8 for the penny spinning experiment) will lead them to the conclusion that a larger sample is desirable. Quality control, and similar statistical concepts can be discussed with this much background, but, perhaps, should be saved for a later time.

Section II - Chapter 5

MECHANICS AND SLOPES

Prologue:

Among the applications of mathematics, mechanics is one of the oldest and most fruitful. The phenomena modeled by mechanics are all around us. The desire to understand them strongly motivates the development of the model and the requisite mathematics. Mechanics is not normally introduced into the curriculum until late in high school. Below, we discuss units, for use throughout the elementary school grades, which experimentally introduce statics and Newton's Laws. The necessary analytical understanding and skills are developed in parallel. Mathematical models of the experiments are eventually evolved, and predictions made and tested. The prediction of parabolic motion in a gravitational field is the culmination, in grade six, of combining models based on experiment with the algebra and simple calculus developed over the elementary school years. The ability to predict a free flight trajectory should add greatly to the sense of achievement of the student, as an addition to his intrinsic intellectual interest in matters of analysis.

Experience in the classroom has shown that a proper appreciation of the physics and mathematics needed for the trajectory problem does not permit one to treat this topic in isolation. A partial discussion of motion in a gravitational field and the slope of a parabola had some success with students who had a conventional education through sixth grade, but the drawbacks were obvious. As the need arose, it was necessary to teach graphing, the multiplication properties of negative numbers and of fractions, and elementary algebra. This interrupted the flow of ideas concerning limiting slopes and the solution of the equations of motion. Interest and attention were frequently lost because of the interruptions. In addition, a careful development of, for example, the early phases of the treatment of fractions was impossible because of lack of time.

Similarly, to extract Newton's Laws, one had to resort to a quick extrapolation of the students' previous experience, plus a few crude experiments with the sliding of chalk and blackboard brushes across the floor, and the tossing of chalk through the air. It is, of course, better to let the students build up experimental techniques, discover the hypotheses, check them to sufficient accuracy, and test out alternate hypotheses.

The sequence of the following units is, it is hoped, consistent with the needs of other parts of the curriculum for developed mathematical and physical insights and skills. All the major mathematical material is of such general utility that its early development would be thought desirable even in the absence of mechanics in the curriculum.

The road to free flight presented here is a possible one, but is not unique. Students prepared for algebraic manipulation by a different course of study, or aware of Newton's Laws through a different set of experiments, may still analyze the trajectory problem as described below for the fifth and sixth grades. The material presented for the earlier grades is given as an example of a curriculum which would permit the teaching of the fith and sixth grade units and also be consistent with the present overall viewpoint of CCSM.

Grades K-2

In K and 1, the child should develop familiarity with geometrical figures, with numbers, with measurement, and with balances and balance boards. The first approach should be playful and open ended so that intellectual interest can be awakened and an intuition based on experience developed. A liberal amount of material such as the Miss Mason School project for CCSM, Marion Walter's mirror cards, and some ESS units on the balance and special blocks, should be introduced.

By the end of the first grade, a beginning can be made on the structure of real numbers, using the material discussed in the progress reports from the

Lexington project of CCSM. This unit should continue into the second grade. It encompasses inequalities, order relations, addition and subtraction of segments and the properties of these operations, positive and negative regions on the number line, addition of equal segments and the relation of this to the cardinals, the addition of small equal segments to equal a unit segment, and the consequent development of fractions. Overlapping this work on a number line, multiplication should be developed as rectangular arrays, first of dots and later of squares on graph paper. The counting of squares is facilitated by marking the cardinals on perpendicular co-ordinate axes. As described in Andy Gleason's report of his work at Morse School, and also in the May and June, 1965, Estabrook reports, this quickly enables the child to multiply large integers and fractions. Large groups of squares are blocked off, finally in the way suitable for decimal evaluation. Commutativity of multiplication is easily brought out.

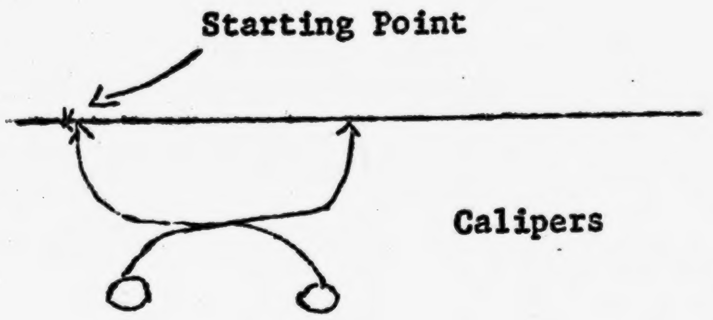
The geometry described by other parts of this summer's CCSM material will include descriptive elements and comparison of shapes and areas, relevant to the work described here. Units on measurement, also being developed this summer will be of importance for the experimental and graphical work below.

In second grade the child can be exposed to the spring. In combination with the balance, results can be obtained concerning the gravitational force, its local uniformity and its dependence on the quantity of material.

The scientist will recognize that the following suggested sequence of measurements implies several important results. It is not intended that the children be instructed to do these and only these experiments. But if they are given time to try things some of them will be done. These can be raised for discussion by the teacher which will lead to questions suggesting other experiments.

Hang a weight on a spring and measure the spring's extension. While oscillations are interesting, a well damped spring may be more useful at this point.

(Relative spring extensions, and for the balance relative distances from the pivot, are all that are of interest here. It is probably better for the student to use calipers rather than a ruler. The caliper distance can then be marked off on a line from a starting point. The first measurement fixes a unit for that spring or student. If, for instance, the subsequent measurement is that of a spring under twice the tension, and the caliper measurement is marked off from the same starting point, the student can readily find with his calipers (or dividers) that the second interval is about twice the first. Ratios such as 2:1, 1:2, 3:2, etc., which will arise below, can be arrived at without requiring reading of a ruler and a subsequent reduction of the ratios to a common form.)



Move the spring and weight around the room, up and down. Does the extension change?

Balance a block by any convenient weights (beans will do). Take another block like the first and see if it balances against the same weights. If it does, see if the blocks balance against each other. Switch pans and see if they still balance. Balance both blocks against beans. How many beans were needed compared to the number of beans to balance one block? If the added beans have been kept separate from the beans balancing one block (by a piece of paper), do the two groups of beans balance each other?

Find two different looking pairs of blocks such that the members of a pair are similar and balance each other. Balance the members of one pair, then add the members of the other pair to each side. What happen?

With a pair of balanced blocks, measure the spring extension for, one, then for the other, and then for both together.

Balance a balance board, which can be distinctly asymmetric. Then balance on it two blocks which have already been found equivalent by balancing against the same amount of beans in a pan balance, or by causing the same spring extension. Measure the distance from the pivot to the blocks. Shift one block, then re-balance by shifting the other and measure again. Shift several times. Having found a third similar block, balance two against one and measure distances from pivot. Shift position of double block and re-balance with single block. This should be repeated for 3 against 1, 2 against 3, etc.

The comparing of weight ratios to distance ratios by tabulating the above data is possible because of the simple ratios involved. In addition rectangles can be formed with the length of one edge representing the number of blocks and the length of the other edge the caliper spread. By comparing areas the above results can probably be extended to less simple ratios.

GRADES 3 AND 4:

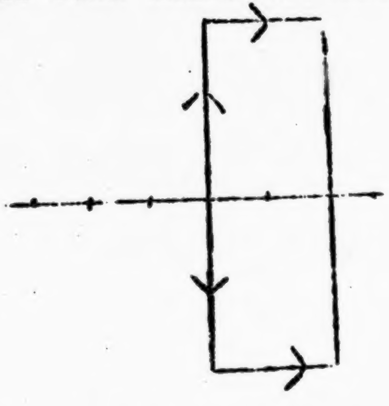
During these grades the important algebraic skills and understanding of the commutative, associative and distributive identities, the properties of 0 and 1, and the addition and multiplication of ordinals and of fractions are to be developed sufficiently for the analysis of grades 5 and 6. A study of sequences will develop the notion of limit. The graphing of pairs of numbers that make a mathematical sentence true will develop graphing skills and the notion of function. More experiments in statics (pulleys and force tables) and a beginning of dynamical experiments (objects rolling and sliding on the level, bouncing from walls, and falling through air) are to be introduced in these grades.

It is recommended that a discussion of true, false and open sentences and identities, as in the Madison Project, be given a prominent role in third grade. A good sense of variable is given by the use of frames into which numbers can be inserted. The method permits a trial and error approach to finding number pairs or number n-tuplets (depending on the number n of different frames) which make a sentence true. For non-identities a functional relationship is implied between the numbers of the n-tuplet, for which a graphical representation is to be developed. This also leads to the identities associated with the properties of 0 and 1, and the commutative, associative and distributive laws. These should be discussed graphically as well. (The associative law for multiplication requires a 3-dimensional construction).

The effect of adding constants to both sides of an equality, or of multiplying both sides by the same number, should be discussed in terms of maintaining a balance between the sides. Then this philosophy should be checked by comparing the "true pairs" (or true n-tuplets) before and after altering the form.

Multiplication of signed numbers should first be discussed in the context of physical situations which indicate the "natural" conventions for multiplication of signs. One situation found to be natural and convincing, though a little laborious, uses rate x time: at this moment I am at zero on a number line, a) if I am travelling forward at 3 mi./hr. where will I be two hours from now, b) where was I two hours ago; c) if I am travelling backward at 3 mi/hr., where will I be two hours from now, and d) where was I two hours ago?

It has been found profitable to exploit the fact that multiplication has been introduced as rectangular arrays or areas. Marking off the areas on cartesian axes for 3 x 2, (-3) x 2, 3 x (-2) gives congruent rectangles appearing in different quadrants. It has usually already been accepted by the student, from a wealth of examples, that (-)x(+) = (+)x(-)=(-) are the convenient conventions. Thus the opposite second and fourth quadrants are seen to imply the same sign. Does the third quadrant then give the same sign as the first? Attaching a "sense" to the area has proved of great interest to students. They have treated 3 x 2 as an ordered pair, represented by, for instance, 3 up and 2 across. In that case while constructing the rectangle they are moving in a clockwise direction around it. Similarly, (-3) x 2 indicates counter-clockwise, etc. The fact that (-3) x (-2) circulates just as 3 x 2 is impressive to the young student. Later they will see that the choice (-) x (-) = + also has the property of keeping straight lines straight, even when they have negative slope. They can also check the distributive identity, and see that it is maintained for negative numbers only with the above choice.



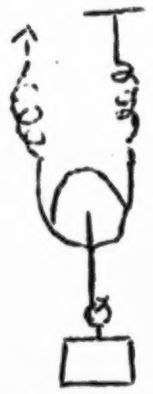
It has proven to be more difficult to teach a proper understanding of operations with fractions, and more time is required for teaching this. The addition of specific fractions on the number line and their multiplication on two dimensional rectangular co-ordinates leads to accurate results easily. In this way, one illustrates in these particular cases that the distributive, commutative and associative laws apply to fractions. It is more difficult to generalize the particular results to the forms of the operational algorithms.

Subdividing the number line brings out the technique of least common denominator to add fractions. Similarly the equivalence of 2/4 (=2 x ¼) to ½, etc., can be shown graphically. The denominator denotes the number of equal subdivisions of the unit segment, and the numerator denotes the number of these added together. With the same representation on crossed axes, the subdivisions of the unit square (or rectangle or parallelogram, if flexibility is desired) can be seen to lead to the rule of "multiply the numerators and the denominators". For an able class, this can be supplemented by an axiomatic approach, as described in the 1965 Estabrook progress reports. The above approach to signed numbers is also reported on, once for third and fourth grades, and again for fifth graders.

With respect to the mechanics experiments, a quick review of K-2 results with springs and balance arms should be supplemented by experiments with several weights poised on both balance arms. In this way, the student should arrive at the explicit law of the lever for parallel forces (gravity); $\sum_i w_i x_i = 0$, where w_i is the weight of each object at a co-ordinate x_i along the lever arm from the pivot. Combined with the pulley, whose use is described below, the law can be verified for vector (one-dimensional) forces by arranging to pull up on the arm. As in Ed Prenowitz's ESS work, pivot points should be available such that cases of stable, neutral and non-equilibrium arise. The students can discuss reasons for these three cases.

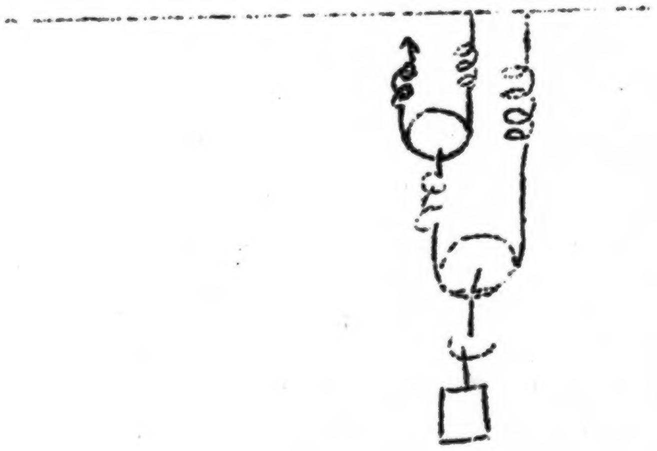
A free pulley is an example of a balance of forces (there is no unknown constraint such as at a pivot). At the same time one learns that constant tension is transmitted along a light rope and a class of simple machines is discovered. With springs inserted along the rope the following experiments can be done.

Find the extension of each of two springs when a given weight and pulley is hung from them. Then insert them in a rope on each side of a simple free pulley. Hang the weight from the pulley. The weight should be substantially heavier than the rope and springs. Measure the extension of each spring. Repeat with several weights.

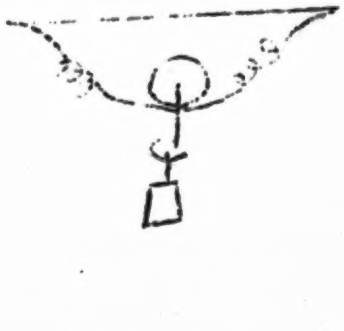


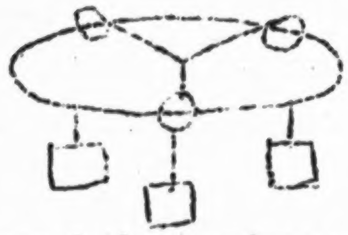
To further illustrate the transmission of tension, the springs can be inserted in any rope and pulled arbitrarily. The spring extension will be roughly proportional and in the same proportion as obtained when hanging the same weight from each of them.

The class should then be given time to find ways of lifting the weight with less than half the force required to lift it directly. They should be supplied with equipment which permits them to put together systems such as that in the figure.



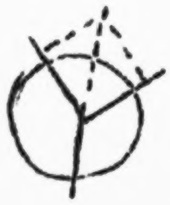
Non-vertical ropes as illustrated may lead the class into questions of forces not in line with each other. Work with "force tables" can elucidate these questions



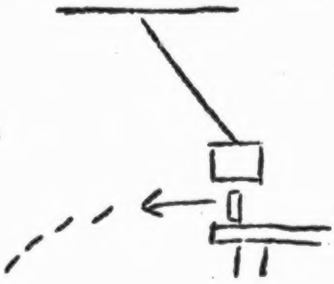


In fourth grade the class may have had experience in the composition of translations leading to vector addition, as in the symmetry unit of the 1964, 1965 Estabrook project. In that case, it would be natural for the class to consider the components of the forces, using geometrical projection.

--A force has a magnitude and direction, like a vector. Is it also the equivalent of its components, i.e. two forces with magnitude and direction of the components? This can be checked as in the diagram. Balance two weights (they must balance on equi-distant lever arms to balance on the force table). By dropping perpendiculars to right angled axes marked on the table from a unit distance along the direction of one rope, and multiplying the projections by the number of units of the weight, one obtains the components. The weight on that rope is removed and one can try to balance the remaining weight by weights pulling along the axes. When balanced one can compare to the components. Then balance 3 weights in arbitrary directions and add up components in two orthogonal directions. Different orthogonal axes should be tried for the same set of weights.



It is also worthwhile to test the independence of components of force roughly with respect to the horizontal part of the motion in free flight: Knock the piece of chalk out horizontally, which releases the pendulum at the same time. Count the swings of the pendulum until the chalk hits the floor. Then knock out the chalk with much greater speed horizontally. Are there more swings of the pendulum before the chalk hits the floor? Gently push the chalk out so that it falls almost straight down. How



many swings do you expect now? Discuss the result with respect to the independence of horizontal and vertical motion. In which direction is the force acting? Does the vertical force change with increased horizontal speed? What determines how long the chalk takes to hit the floor?

The question should then arise as to what happens if forces are not in balance. Then things move. Do things ever move when there are no forces on them? The sliding of blackboard brushes on the floor, followed by the rolling of balls, etc., would lead to a discussion of whether the slowing down is due to a frictional force, or if the slowing is there in the absence of forces. The children can suggest ways of decreasing the friction such as greasing a piece of chalk before sliding it. To compare the slowing down of objects that start off at about the same speed one can slide them down a steep inclined plane for a start, or give them an impulse from a rod on a spring (as in a pinball machine). By these means the class can reasonably conclude that when the forces on an object are removed, it moves nearly in a straight line and keeps up its speed for a long time. The conjecture that with no force it would move uniformly would not be amiss, but should not be considered as proved, but only to be a working hypothesis.

In the fifth or sixth grade these students will work with a dry ice puck. With the very small friction involved in that case the above hypothesis will work within experimental accuracy.

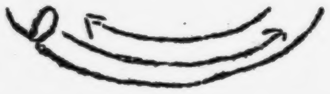
GRADES 5 AND 6

In grade 5 the investigation of motion without force should be extended to the dry ice pucks on glass plates of PSSC as mentioned above. The timing as the puck goes by uniformly spaced lines on a long table can be made with a series of stop watches, or by marking paper tape as it is pulled uniformly, perhaps by the puck itself.

By marking vertical lines on a wall it should be possible to verify approximately the uniformity of the horizontal part of the motion of an object in free flight.

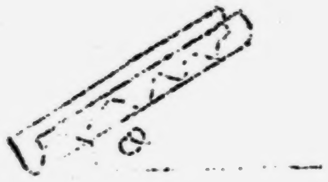
It is then time to investigate the behaviour under a uniform force such as gravity. In questioning the students it arises that many believe that a constant force increases the velocity at first, but then the effect saturates at some velocity dependent on the force. That this is not a ridiculous idea is illustrated by the known effect of forces as v approaches the velocity of light. Indeed, fluid frictional forces bring about a terminal velocity, as they come into equilibrium with the applied force. Experiment is clearly required to separate out possible hypotheses.

The following "inclined trough" experiment has been suggested by Steve Willoughby. The use of a trough instead of a plane allows the timing of equal intervals, with accuracy, by the periodic oscillations from side to side. With the trough in a horizontal position, hold a small steel ball up against one side and release. The number of oscillations in 5, 10, 15, 20, and 25 seconds can be counted. The same procedure can be repeated with the ball released from a higher or lower position.



trough, end-on

If the students have had the ESS pendulum unit they will not be surprised at the constancy of the period of oscillation with respect to amplitude. In any case, this is to be accepted as a timing device, not to be understood in detail. The vertical motion to be investigated will be assumed to be independent of this sideways motion. The period of sideways oscillation is not independent of the



tilt (it increases as sec θ) because "sideways" changes its orientation with respect to the vertical. The period must be remeasured for every angle of tilt used. Once having established the period, the rate of motion "along" the trough can be investigated. A piece of carbon paper over white paper (or simply a piece of impregnated paper) is laid smoothly in the trough. The ball is released from the side, near the top end of the trough, and traces out its path as it oscillates downward. To establish the uniformity of the acceleration one need only compare the differences of the distances between successive pairs of nodes (or maxima, whichever is more convenient). These increases in distance between nodes should be roughly constant. Discussion is required to relate this to a constant increase in the average velocity between nodes. As the node spacing represents equal time intervals, this implies a constant increase in the average velocity in each unit of time. The increase in velocity in feet/second each second can be computed, and will depend on θ. By observing the trend of this acceleration with θ, the students can find a lower limit to the vertical acceleration, and probably extrapolate to within a factor of two of the correct result. Because the rolling motion has through the rolling constraint force, transferred some energy into the balls rotation, this acceleration is not g. It is $\sqrt{\frac{5}{7}g}$ for a uniform sphere which probably will not be experimentally distinguishable from g.

The students have now been able to hypothesize, and check to a reasonable extent, Newton's First and Second Laws (they do not need the Third Law for the work described below) if one ignores the rolling constraint. An experiment in which the constraint is not hidden, is to drop a weight which is pulling a paper tape through a PSSC bell clapper timer. Depending on classroom experience this latter experiment can be used to supplement or replace the inclined trough experiment. In either case, one should use several balls or weights to arrive at the independence of the acceleration from the object's mass (when rolling, the mass distribution matters).

In fourth grade, in investigating true pairs, the students have probably plotted some straight line graphs and other functions. Then, or now in fifth grade, they could also plot e.g., the height of a semi-circle against the length of the perimeter to that point and obtain the trigonometric functions experimentally. Other functional relationships can be explored graphically. A unit of this type is described in the "functions" section of this conference report.

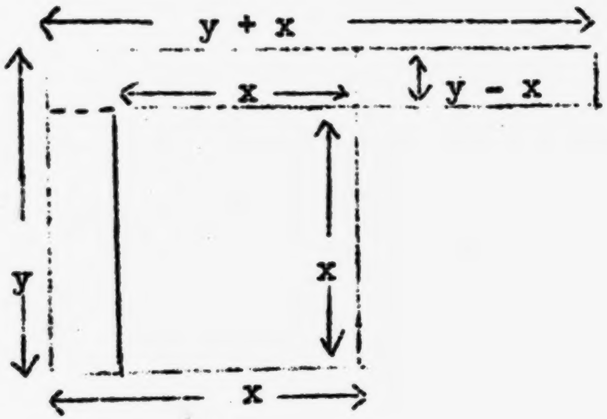
In the fifth grade the details of the linear function graph should be investigated as in the Morse School and Estabrook projects. The special motivation in this context is the need to have an expression for the motion of bodies sliding along the floor. Is the slope (increment up divided by increment across) the same between all pairs of points on the line? What part of the mathematical sentence determines the slope? Find a sentence whose graph has a very steep slope. A very small slope. A negative slope. A horizontal line. A vertical line. How can we get sentences whose graphs have the same slope but are displaced parallel to each other? Given these two points, what is the slope of the line joining them? What is its mathematical sentence? What is the sentence of a line with this slope going through this other point?

The introduction of sequences was suggested for the third and fourth grades but not discussed further. The following can be started in fourth grade and continued in fifth grade.

The series $1 + 3 + 5 + \ldots + (2n - 1)$ can be summed geometrically as a series of "wrappings" making up an $n \times n$ square array. This is discussed in the Estabrook 1964 reports for third and fifth grade. To continue with the idea of mathematical induction to an algebraic proof of the above sum is optional to the following material. A discussion of a surface wrapping about a cube in a three dimensional corner, would be a valuable extension of the geometric reasoning involved in the plane; leading to a series that sums to n^3. How do these sequences behave for large n ?

They can discuss the sequences given by $1 - 1 + 1 - 1 + \ldots + (-1)^{n+1}$, 1/2, 3/4, 4/5, $\frac{n}{n+1}$, with respect to upper and lower limits for large n. The harmonic series can be grouped to show that it diverges. They could check the closed form of the sum of a geometric series by substitution for many n, and then look at the large n behavior.

In the sixth grade one can note that thrown objects do not move in straight lines and that some familiarity with mathematical sentences whose slopes change is required to be able to handle this aspect of mechanics. The fifth grade 1965 Estabrook project and the Morse School 6-7 grade summer project can be followed. This develops the limiting procedure of successive chords to obtain the tangent to a parabola at a point. In addition to the algebra previously discussed they must be able to factor $y^2 - x^2$. This can be done by using the distributive law twice on $(y - x)$ $(y + x)$. It can also be done geometrically



The Estabrook "slope" unit should be preceded by a careful discussion of nested intervals and successive approximations as indicated by the Function section of this report.

The previous classes have only had time to find the slope at a given point. This can be extended to the slope at a variable point in a few more sessions.

Using the First and Second Laws of Newton one can go through the following algebraic steps, using a horizontal speed of say 2 feet/sec, and g = 32 feet/sec/sec. At every step many time pairs are to be found, to give a well understood meaning to the mathematical sentence.

From Newton's First Law

□ = horizontal displacement in feet = 2 x ○ where ○ = time in secs.

△ = vertical displacement in feet = ◇ x ○

where ◇ = average velocity in feet/sec. (average of initial and final velocities)

The use of average velocity is frankly taken as an estimate at this time.

If ▱ = final velocity in feet/sec

then ▱ = 32 x ○ from Newton's Second Law and experiment.

Then ◇ = (32 x ○ + 0) /2 = 16 x ○.

Then by substitution (which is an algebraic device that requires discussion and checking)

△ = 16 x ○ x ○ = 16 x 1/2 x □ x 1/2 x □ = 4 x □ x □.

Thus the average velocity "approximation" gives the parabola. It now remains to refer back to the tangent result to show that the rate △ changes with ○ is 2 x 16 feet/sec/sec. at every instant. Thus, Newton's Second Law is exactly satisfied.

If aparabola is drawn on the blackboard it is not difficult to throw a piece of chalk so that it follows it closely. If one graphs a family of parabolas with a common vertex, then it is easy, by starting the chalk horizontally at the vertex, to closely follow one of them. While looking by eye is inaccurate point by point, the following of the whole length of the curve makes a fairly accurate experiment.

CIRCULAR FUNCTIONS

During the early experiences with measurement, children will be given several wheels of differenct sizes to measure. These could be bicycle wheels, wheels made out of several layers of corragated cardboard (in which case spokes should be drawn on the cardboard, and there should be the same number of spokes on each wheel), etc. They will be asked how big is the wheel, and be expected to come up with many different answers. Differences in answers will come from the fact that different things will be measured.

One set of measurements which the children would be encouraged to get would be the lengths (to the nearest unit) of many chords. They would note that there seems to be a largest one (namely, the diameter) which would give a feeling for least upper bound. Strings could be attached for chord measurements.

They would also measure the radii. Within their limits of accuracy, each child should find that all the radii of a given circle have the same length. The children would be asked to find out how "far around" the circle is, and presumably would try to wrap a string around the wheel, and also roll it along a straight path (marking the starting place on both wheel and floor) and measure the length. There are also various other methods which they might try, such as measuring small chords, etc.

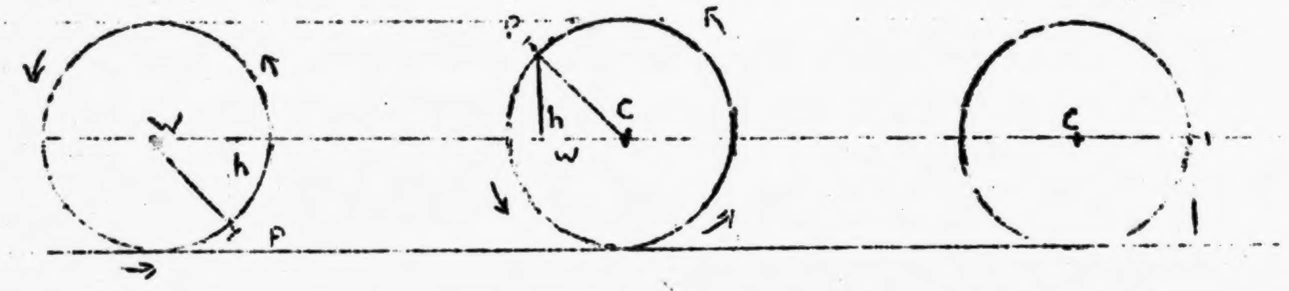
The children would be asked to place the centers of the circles together (holes through the centers would make this an easy task) and notice that the spokes "line up." Then, one of the circles could be rotated to notice that the angles (or spoke spaces) are congruent and that there are the same number on each wheel even though one wheel is larger than the other -- thus, getting a feeling for congruence of angles. Arc length for a particular spoke space, or a particular number of spoke spaces would be measured (roll the wheel or wrap the string) for the various wheels, and compared with other numbers also, to begin developing a feeling for radian measure. The total angle (number of spoke spaces) for each circle would also be noticed.

Area of the circles would be approximated by placing a grid (on transparent plastic) over them and counting the number of squares. Now, the various measures for the circles would be compared and it would be noted that there is a linear relationship between the various one dimensional measures and a quadratic relationship between one and two dimensional measures -- these would be found from the table. For this purpose, it would probably be desirable to have the ratios of the radii of the three different sizes of wheels be 1:2:3.

COMPARISON GAMES

Late in the elementary school program, the wheels would be/ used again to develop the sine and cosine functions. For this unit, it would be desirable to have a wheel which has radius of one decimeter, and use a meter stick and tape for all measurements. A long sheet of paper (brown wrapping paper, or butcher's paper, or something of that sort) would be taped to the wall with masking tape, and the wheel would be rolled along the floor with a pencil through the hub so that a horizontal line can be drawn at the height of the hub. If there is a significant base board in the classroom, it may be desirable to use a long (at least three meters) piece of plywood as a backing for the paper.

Now, attach a metric tape measure to a point on the edge of the wheel, start the wheel in position 1, and roll it to the left. After the wheel has moved a given distance, say ten centimeters (this figure can be found by looking at point of the meter tape that is on the hub line), mark the center point with a colored pencil, and mark the position of point P with the same colored pencil, Continue doing this for various points. Then, draw line segments (corresponding to radii) between corresponding center points and P's; and measure and record, in tabular form, corresponding arc lengths (a), heights of P above the hub line (h), and horizontal distances of the center from the perpendicular projection of P onto the hub line w. As the dots are being made, the value of a could written beside each position of P.



After a dry run, in which the pupils would make only a few measurements, the class would be divided into small teams, and collect data for values of a between 0 and 125 centimeters which are multiples of 2 centimeters. Each team could work on this while the other pupils were working on some other subject so as not to waste the entire class's time. It would be agreed in advance that values of h would be positive above the hub line and negative below, while value of w would be positive if P is to the right of the center and negative if P is to the left of the center.

Tables would be made to compare each of the following: number of spoke spaces and arc length; arc length and h; number of spoke spaces and length; arc length and w; and number of spoke spaces and w.

Discrete graphs would be drawn for each of these five tables, and the pupils would then try to guess intermediate values -- checking by going back and making the appropriate measurements.

Then, the pupils would be asked to look for various interrelations among the functions. They would presumably notice that both the sine and cosine functions are periodic and that the cosine is ¼ of a phase behind the sine function. They might then be asked to square each value of the sine and cosine and look for a relationship. Presumably, they would notice that the sum seems to be relatively constant and would be asked whether they think the variations are measurement errors, or more essential than that. Then, they would be asked if they could give a convincing argument for the fact that $\sin^2 x + \cos^2 x = 1$ for any value of x (using their knowledge of the pythagorean relationship).

Before the paper is destroyed, it would be worth noting that the dots on it are points of another curve known as the cycloid. It might be possible to have some pupils construct a curve of quickest descent using cardboard and marbles to test.

Numerous physical examples of sine functions would be contructed by the pupils at this time. For example, a circular trough (similar to the trough used in the physics experiment to demonstrate Newton's second law) would be placed horizontally and a ball would be shot into it near the top of an edge. The path traced would be essentially a sine curve (or a dampened sine curve).

Another experiment could be done with a pendulum with a long period. This can be achieved by attaching a long rope to the top of the gymnasium and having a child swing on it. Marks would be made on the floor at equal intervals, and and the time at which the child crosses each mark would be recorded, and later, the distances would be plotted against time. Again, the result should be a dampened sine curve.

If the average mean temperature in a particular city for every day of the year can be acquired from the weather bureau, the graph of these will approximate a sine curve also -- the process of determining where the origin is may take a few minutes, but is an interesting process. (By mean temperature, we mean the average of the high and low temperatures for the day, and the average mean temperature is the arithmetical mean of these over a long period of years.)

Other examples of sine curves (or approximations to sine curves) can be found in many places. The path of an earth satelite is one example, alternating current can be used to generate an example, various osciliscope type machines are used in some garages to test automobiles and are expected to produce sine curves if the automobile is healthy.

Various functions would be studied, including the mapping of arc length (s) onto height of P (h) (the sine curve); s onto a (angle measure -- number of spokes could be replaced by number of degrees, etc., to get still different functions): s onto r (length of radius); and s onto w(cosine). The domain would be changed so that it included only arc lengths from 0 to 2 at first, and then this would be increased to include larger domains, finally including the entire range of real numbers, positive and negative (roll the wheel backward).

Composition of functions would have been considered earlier, but would be reconsidered in relation to these functions. In particular, the composition of the functions from a to s, and the function from s to h would be used to produce a different sine function which has as its domain the set of angle measures rather than arc lengths, etc.

The question of which functions have inverses would be discussed, and the pupils would be asked to decide what the domain should be in order to have a sine function that has an inverse function (and similarly for cosine).

Now the sine and cosine functions would be studied in considerable detail, with the children explicitly mentioning all symmetries they could find including the translational symmetry of periodicity. Even and odd functions would be studied and other functions which are even and odd would be found by the pupils. These would include the obvious polynomial functions, the absolute value function, and any other function they might discover (a graph without an explicit algebraic formula would be entirely acceptable in this regard, though it might be fun to try to find an algebraic rule). From all of this information, the children would be asked to decide how much of the table for (e.g.) the sine function they would need in order to construct a sine function whose domain is the set of all real numbers.

Next, the question "What is $\sin(x+y)$?" would be asked. The obvious answer of $\sin x + \sin y$ could easily be shown to be wrong by using simple counterexamples (the children can provide these easily). Then, using their knowledge of coordinate geometry (including the distance formula) they would graph points at a circular distance of y, x, and x-y from the point (1,0), find the lengths of appropriate chords, set them equal, and see what happened. Using the fact that $\sin^2 z = 1$, they could easily derive the usual formula for $\cos(x-y)$. From this, with a few simple algebraic manipulations, they can derive the formula for $\cos(x+y)$, the usual relation between sin x and cos x(which they will have suspected earlier), and the corresponding formulas for sin. Then, the double and half angle formulas would be derived.

Now, it is time to use this new found power to construct a better table. The values of sin , cos , sin , etc., can be determined with complete accuracy using the pythagorean relationship, and the formulas mentioned in the above paragrah can be used to find the values of any numbers which can be written in the form $\sin \frac{n\pi}{2^p}$, or $\sin \frac{n\pi}{3 \cdot 2^p}$, etc. Thus, if a value of π were known with sufficient accuracy, it would be possible to determine the value of sin x to any desired degree of accuracy for any x. Some values of a table should be calculated in this way (e.g., $\sin \pi/8$, $\cos 22\frac{1}{2}^\circ$, etc.).

The pupils will now have constructed two different tables. One based on their original measurements with the rolling circle, and a second based on theory. It can be pointed out that there are still other methods of calculating the values for such tables, but infinite series should probably not be considered in detail at this time. Now, it would be appropriate to give them tables which have been made up by others. This should include tables with a domain of real numbers (or arc measure -- the tables look the same even if the functions are technically different) as domain, and tables with angle measure in degrees as domain.

Then, as a theorem, it can be shown that the traditional formulas sin θ and cos θ where θ is an acute angle of a right triangle are ture, and applications involving right triangles can be considered. Included in this would be explicit consideration of the inverse functions. During this study, it would become clear that another function (namely the tangent function) would be very convenient, and tan x would be defined in terms of sin x and cos x and studied first as a function from the reals to the reals ("Is tan even or odd?" Is it periodic?" "What is an obvious difference between it and the other two function?" etc.). A very informal discussion of limits might be appropriate at this time. Then, of course, the tan function would be used to do some right triangle trigonometry.

Work with oblique triangles and trigonometric identities and equations should probably be saved for a later time -- presumably somewhere in the junior high. When identities and equations are considered, the pupils would be expected to mention quite explicitly any restrictions on the domain.

Circular functions will be reconsidered again in later grades in connection with complex numbers, vectors, and analysis, as well as the usual topics mentioned above.

Section IV

WORKING PAPERS ON GEOMETRY

Most of the man-power, and all the woman-power of the conference was concentrated on geometry. This, and the inclusion of brief outlines together with units described in some detail, has resulted in the fifteen working papers presented in this section. These papers were not written for sequential presentation, nor does each take into account all the possible interactions with the other units. We believe they are all sufficiently consistent with each other to be in one curriculum, and that they indicate a large part of the coverage such a curriculum should have. Some of the units rely on the earlier teaching of other material as indicated in the paper. Some papers or subsets of papers are nearly independent of the rest.

For the above reasons the order in which these papers are presented is somewhat arbitrary. We have attempted to put those units which start in the earlier grades before those meant to start in later grades. There is still much overlapping in grade level between papers, as some units that start in K or 1 end in grade 6 or later. In general the fine grained choice of grade level and ordering of material has been left to be decided by experience.

Chapter 7

EXAMINATION AND DESCRIPTION OF COMMON OBJECTS

An assortment of "standard" physical objects (cube, ball, cylinder, etc.) of various sizes and colours is used.

The physical properties are discussed (e.g. flat-round, size, hard-soft, number of vertices, etc.)

Discuss which properties could be determined if the objects were in a cloth bag, which could be felt but not opened.

Games of the following sort should be used:

1. Attribute blocks
2. Fitting blocks in holes
3. Guessing games with one person putting an object in the bag and answering questions about its physical nature - This is an exercise in abstract visualization.
4. Games in which partners describe unspecified objects on paper - exchange papers - and then guess what object was meant.

Chapter 7

EXAMINATION AND DESCRIPTION OF COMMON OBJECTS

I. Pressman

Rewrite M. Walter

I don't think there is anything new here that is not available already. But anyway, here is a shortened version. I think Attribute Blocks is the only "new" thing.

Properties (such as flat, round, hard) of objects (such as a cube, ball, pencil, glass) of various sizes, shapes and colors are discussed. Which properties could be determined if objects could be felt but not seen? Games such as:

"Attribute Blocks"*
Fitting blocks into holes
Twenty Questions

can be played.

*Available from ESS

Comment by E. Lomon: I recommend that in the final version Bill Pettis should include Irwin Pressman's remarks in Chapter 1. Then Chapter 7 can be omitted and all later chapters re-numbered. Marion Walter's condensation indicates how it can easily be included in Chapter 1.

Section IV - Chapter 8

PLAYING WITH FIGURES, BLOCKS, AND TESSELATIONS

Assumption: The children know the distinguishing features and the names of the the triangle, square, hexagon, pentagon, n-gon, regular n-gon, quadrilateral (4-gon), etc.

I. Blocks

The blocks considered are the set constructed by E. Prenowitz. Some time will be given for familiarization through play.

1. The children are asked to make a square using 4 squares. Next they are asked to make a larger square.

Can a square be made out of 2, 3, or 5 squares? Give reasons.

Do the same thing with triangles.

2. Make a hexagon of two (red) quadrilaterals, 3 (blue) parallelograms, and 6 (green) triangles. Surround.

Make a large hexagon using all sorts of pieces - Can it be made larger (i.e. by using more pieces)? Introduce the idea that by bringing in more pieces the figure gets bigger - and the only limitations are the number of pieces, size of floor, number of workers, etc.

Create a chip trading game by noting that from the comparison of sizes

1 yellow = 2 red 3 blue = 6 green

At a later time assign some value to the orange square - let the children choose - (or let 5 yellow = 1 orange). These might be used instead of Cuisenaire rods.

3. Regular n-gons. Ask the children to create regular 9-gons (1 triangle, 3 squares on the edges and then fill in with 3 parallelograms). Trace this on the paper.

The regular 12-gon next. (hexagon at center, 6 squares, then 6 triangles) Trace this on paper.

Make a 5-gon and a 7-gon. Let the children convince themselves that regular 5-gons and 7-gons cannot be made with this set.

What is the largest square that can be made with this set?

4. Create symetric patterns of blocks. Consider various types of symmetry

a) Symmetric about a line (180 degree symmetry)

b) 120 degree symmetry

c) 90 degree symmetry

d) 60 degree symmetry

Decide which patterns are left unchanged by putting 1 or 2 mirrors along the line (s) of symmetry. (In the above cases all may be obtained - it should be demonstrated to the children that if they make a symmetric pattern and set up the mirrors then all the blocks behind the mirrorscould be removed.)

5. Tile the plane using only 1 type of block at a time (square, triangle, parallelogram, hexagon). Trace these and keep.

Now try to do it with all the blocks together.

Encourage the children to relate the traced diagram with the actual configuration of blocks.

II. Paper Cutting and Pasting

1. Cut a large number of "congruent" paper triangles and arrange them symmetrically on a plane. Obtain 60°, 90°, 120°, and 180° symmetry.

Now tile a page with these triangles (paste them down if necessary).

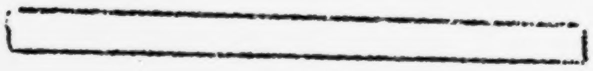
Repeat this with irregular congruent quadrilaterals. Tile the page with these also - This can be done.

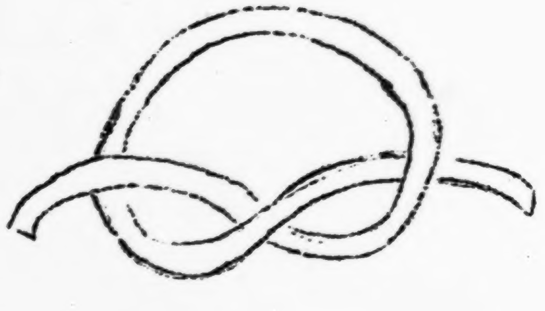
2. Cut up a rectangle to make a triangle.

Cut up a triangle to construct a quadrilateral

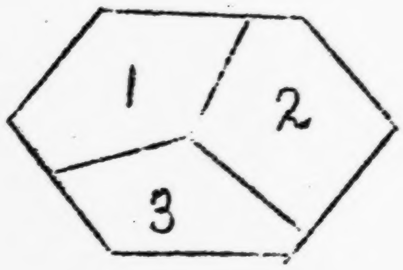
3. Ask the children to cut as good a regular pentagon as possible. Teach

them the following technique: take a long thin strip of paper, tie an overhand knot, pull it tight slowly, and behold a regular 5-gon!





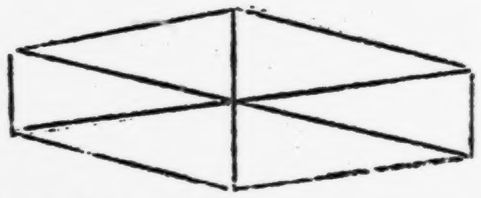
4. Cut a congruent regular hexagon into 3 irregular congruent pentagons as shown in the diagram. Then persuade the kids to tile with these.



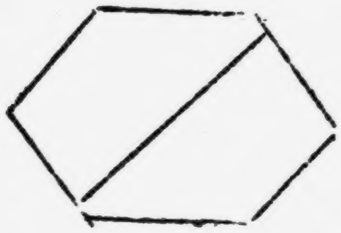
Next cut arbitrary irregular pentagons and try to tile. Indicate that these don't work because there need not be a way of filling in all the region about each corner.

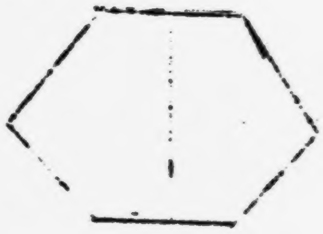
Try this for 7-gons too.

5. Tile with "stretched" hexagons



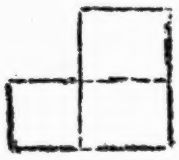
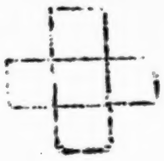
These can be cut into 2 equal pentagons in various ways - and one can tile with them also



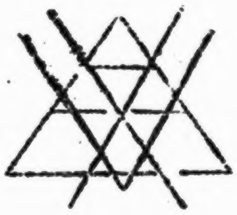


etc.

6. Get children to solve tiling problems on graph paper - without cutting and pasting from a different sheet -

e.g.  ,  and what they will.

Do the same with the Chinese checkerboard configuration.



7. Make a large square tiling on graph paper. Subdivide each square into 4 squares. Note that this is a tiling also. Repeat again. Let children consider how long this process might continue (i.e. until the physical dimensions become impractical).

Repeat with triangles.

8. Cut a long string of paper dolls (cut several strings and paste together if needed). Get children to suggest names for the dolls - then suggest numbers - then finally number then ... -3, -2, -1, 0, 1, 2, 3, ... symmetrically.

CONSTRUCTIONS

Considerable time and effort should be devoted to the development of intuitive geometry at all grade levels. Students should have many experiences with physical aspects of geometry long before any formalization and abstraction of geometry is attempted. The formal study of geometry should serve to organize and to structure concepts of geometry, many of which are (or, ought to be) present in the students' stockpile of intuitions about geometry, in particular, and about mathematics in general. To gain these intuitions, the students should look at and handle objects of various shapes to see "how they are put together", and they should learn to make models of and replicas of common objects, and they should learn to use compasses, straight edges, pencils, and the like to draw pictures of objects and shapes. Much of this work involves what can be thought of as construction.

Geometric constructions provide much substance for applications in geometry as well as for motivating the study of geometry. The term construction in this context is to be interpreted very broadly, and includes, in addition to the standard uses of straight edge and compasses, such notions as folding, cutting, and posting of materials as well as replicating, molding and drawing objects. So, constructions should be an integral part of both informal and formal developments of geometric notions at all grade levels.

Early Period. At the earliest stages, any K-2, construction activities might be classified in the categories (a) pattern-building activities, (b) replication and modeling activities, and (c) cutting and folding activities. Among the activities in (a) are building with blocks, making tiling patterns with blocks (both regular and irregular shapes), making wall paper designs by posting polygonal shapes on rectangular sheets of paper and for cylindrical tubes, putting together jigsaw puzzles, and making beads-on-wire designs. From these activities, the students should be expected to gain some familiarity for for the "feel" of various geometric shapes as well as for how certain shapes "fit together" well while others don't. In particular, the students should gain an awareness of the fact that a plane can be tiled with copies of an arbitrary triangle and, also, with copies of an arbitrary (plane) quadrilateral. The natural extension of this notion to try to build an awareness for "filling up" space is almost too inviting to avoid. The students should be able to verify experimentally that space can be filled up by, say, cubes (the 3-dimensional analogue of the square) but not by regular tetrahedra (the 3-dimensional analogue of the regular triangle).

The activities involved in category (b) include making models of figures from clay, cardboard, wire and string, making facsimiles of coins (and other raised patterns) on aluminum foil, making figures with line symmetry using ink blots, and replicating figures through the use of carbon paper and potato block printing. As with the activities described earlier, the students should be expected to become familiar with various geometric shapes and how these shapes are"put together".

At the early stages, the activities of category (c), cutting and folding, should be of very simple sorts. As a most elementary observation, the students should note that when a sheet of paper is folded in half, the edge of the fold is straight. As a contrast to this phenomenon, they should observe what happens when a "non-flat" surface, such as irregularly stretched crepe paper, is folded. That the edge of a fold in a sheet of paper (essentially a model of a plane) is straight illustrates an important theorem about intersecting planes. This allows one to make a straight edge from a sheet of paper. Among the kinds of exercises that might be attempted at this level are the following, each involving making a particular fold to meet specified conditions:

1. Make (or, construct) a line or segment.
2. Make a line that passes through a given point. Make several such lines, if possible.
3. Make a line that passes through two given points. Make several of these, if possible.
4. Make a line that passes through three given points (if possible). Make several, if possible.
5. Make two parallel lines.
6. Make several equally spaced parallel lines.
7. Make a line parallel to a given line.
8. Fold a sheet of paper so that (a) one given point falls on another; (b) one given line falls on another; (c) one end point of a segment falls on the other end point.
9. Make use of a folded paper straight edge to (a) compare the length of various segments; (b) draw replicas of a segment; (c) compare the length of a segment with itself "turned around".

Making use of tracing and/or cutting operations, the students can work at the task of producing models of triangles with properties described in these exercises.

10. Make a triangle that will fit back into the hole from which it is cut in only one way.
11. Make a triangle that will fit back into the hole from which it is cut in two ways.
12. Make a triangle that will fit back into the hole from which it is cut in more than two ways.
13. Try exercises 10-12 making a quadrilateral instead of a triangle.
14. If you had two triangles like the one described in exercise 10, in

in how many different ways could you place one of them on top of the other so that the vertices touched? Now answer this question for figures like the ones described in exercises 11-13.

Intermediate Period. In the intermediate period, say grades 3-4, the work with folding and replication continues. In addition, compasses are brought in as construction instruments.

The folding and cutting activities can be extended from folding to make a line to folding a line onto itself to make a square corner. The notion of a square corner gives rise to properties of perpendicularity. Some suggested activities that involve folding a sheet of paper are the following:

1. Make a line that is perpendicular to (makes a square corner with) a given line. Make several, if possible.
2. Make a line that is perpendicular to a given line and passes through a given point which is not on that line. Make several, if possible.
3. Make a line that is perpendicular to a given segment at its mid-point.
4. Given three points A,B, and C, make three successive folds such that A falls on B, B falls on C, and C falls on A. What appears to be the case about the three lines (folds) that are obtained? Repeat with four points and four folds.
5. Given three points L, M, and N, make three successive folds such that L falls on M, M falls on N, and N falls on L. Repeat with four lines and four folds.
6. Fold a sheet of paper in halves, thirds, fourths, etc.
7. Fold and/or cut a sheet of paper to obtain a rectangle, a square, and other polygons.

Having become familiar with various geometric shapes, the motion of line symmetry can be introduced through folding. The shapes (triangle, circle, square, rectangle, heart, ellipse, parallelogram, diamond, kite, egg, etc.) can be folded to test for line symmetry, and can be classified in terms of the number of symmetrical folds. Cutting a folded sheet of paper to obtain symmetrical figures should also be done. (One activity that might prove interesting here is to try to cut a triangle from a folded sheet of paper.)

Figures can be replicated on a piece of paper by folding. Simply draw the figure with a soft-leaded pencil, fold the paper and rub the back to obtain the replica. Doing this with points, segments, angles, triangles, and others, one can notice that the fold is the perpendicular bisector of the segment between any point in the original drawing and its "image" in the replica. Some suggested activities that involve this process of replication by folding are the following:

1. Make a segment that is congruent to a given segment and has a given point as end point.
2. Make a segment that is congruent to a given segment and lies on a given line.
3. Make a segment that is congruent to a given segment, lies on a given line, and has a given point as end point.
4. Make an angle which is (a) congruent to a given angle; (b) congruent to a given angle and with one side common to the given angle.
5. Make an angle which is congruent to a given angle and (a) with one side on a given line; (b) with a given point as vertex.
6. Replicate various polygons with conditions similar to those in 4 and 5.

It is probably feasible to obtain replications of replications by successive foldings in parallel lines (to obtain an image under a translation) and in intersecting lines (to obtain an image under a rotation). Just how much of this can be done in the intermediate stage is not at all clear, but development of intuitions about motions of this sort certainly is a worthwhile long-range objective.

Having done work with folding and replication, it is a reasonable extension to consider the problem of constructing figures when we are restricted to working on a rigid metal sheet or other flat unfoldable surface. For this, motivation must be provided to get to the use of the straight edge for drawing lines and the compasses as dividers for transferring lengths. Suggested activities involving construction with straight edge alone are the following:

1. Given point A, construct a segment that contains A. Construct several.
2. Given points A and B, construct a segment that contains A and B. Construct several.
3. Given points A and B, construct a segment that has A as an end point and contains B. Construct several.
4. Given points A and B, construct a segment that has B as an end point and contains A. Construct several.
5. Given points A and B, construct a segment that has A and B as end points.
6. Given points A, B and C, try to construct a segment that has A and B as end points and contains C. Is this possible?
7. Given segment $\overline{AB}$, construct a segment with end point A that contains $\overline{AB}$. Construct another. Find a few more.
8. Given segment $\overline{AB}$, construct a segment with end point B that contains $\overline{AB}$. Construct several.
9. Given segment $\overline{AB}$, construct several segments that contain $\overline{AB}$, Construct several segments that contain $\overline{AB}$ and do not have A or B as end points.
10. Given segment $\overline{AB}$, construct a segment which meets $\overline{AB}$ only in point A.
11. Given segment $\overline{AB}$, construct a segment which when taken together with $\overline{AB}$ forms a segment. Construct several. Must the segment have A as an end point? Can the segment have A as an end point?
12. Given segment $\overline{AB}$, construct several segments that contain $\overline{AB}$. Can you construct a largest segment containing $\overline{AB}$?

Suggested activities involving construction with straight edge and dividers (compasses) include:

1. Given two segments, test to find out if one is smaller than the other.
2. Given two segments, test to find out if one is larger than the other.
3. Given a segment, construct several larger segments. Is there a largest of these?
4. Given a segment, construct several smaller segments. Is there a smallest of these?
5. Given $\overline{AB}$, construct $\overline{CD}$ larger than $\overline{AB}$. Construct $\overline{EF}$ larger than $\overline{CD}$.
6. Given $\overline{AB}$, construct $\overline{CD}$ smaller than $\overline{AB}$. Construct $\overline{EF}$ smaller than $\overline{CD}$.
7. Given a segment, construct a segment which is twice as large; three times as large; five times as large.

8. Given $\overline{AB}$ and point P. Construct a segment with end point P that is twice as large as $\overline{AB}$. Construct several. The same for a segment three times as large; four times as large.

9. Given $\overline{AB}$ and point P. Construct a segment with end point P that is congruent to $\overline{AB}$. Construct several.

10. Given $\overline{AB}$, line L and point P on L. Construct a segment congruent to $\overline{AB}$ that has end point P and lies on Line L.

11. In Problem 10, how many such segments can you construct?

12. Given point A, find two points B and C such that $\overline{AB}$ is congruent to $\overline{AC}$. Find another pair of such points. Find several points.

13. Given point A and point B, find point C such that $\overline{AB}$ is congruent to $\overline{AC}$. Find several such points.

14. Given point A and point B, find C such that $\overline{AB}$ is smaller than $\overline{AC}$. Find several. The same for "larger than".

15. Given point A, find two points B and C such that $\overline{AB}$ is congruent to $\overline{AC}$ and A lies in $\overline{BC}$. We call A the midpoint of BC.

16.. Given point A, find several segments whose midpoint is A. Can they all lie on the same line? Must they all lie on the same line?

17. Given point A on line L, find a segment that is contained in L and has midpoint A. Find several.

18. Given point A and point B, find point C such that B is the midpoint of $\overline{AC}$. How many such points can you find?

19. Given points A and B, find C such that $\overline{AC}$ is twice as large as $\overline{AB}$. Three times; five times.

20. Given points A, B and $\overline{PQ}$. Find point C on line $\overline{AB}$ such that $\overline{AC}$ is congruent to $\overline{PQ}$. How many such points can you find?

21. Given points A, B and $\overline{PQ}$. Find C on ray $\overline{AB}$ such that $\overline{AC}$ is congruent to $\overline{PQ}$. How many such points can you find?

22. Given points A, B and $\overline{PQ}$. Find C such that A lies in $\overline{CB}$ and $\overline{AC}$ is congruent to $\overline{PQ}$. How many such points can you find?

Later Period. In the later period of elementary school, namely grades 5-6, the use of compasses and straight edge as construction instruments is to be extended. (Even though the use of replication through folding is being "faded out" as a construction tool, some thought should perhaps be given to making use of this process to illustrate the rigid motions of a plane.) Some suggested activities that involve problems about lines and circles are the following:

1. Given line L and point A not on L. Find a point B such that L does not meet $\overline{AB}$. Find another such point C. Does L meet $\overline{BC}$?
2. Given line L and point A not on L. Find a point B such that L does not meet $\overline{AB}$. Find a point C such that L does not meet $\overline{AC}$. Does L meet $\overline{BC}$?
3. Given line L and point A not on L. Find a point B such that L meets $\overline{AB}$. Find another such point C. Does L meet $\overline{BC}$?
4. Given line L and point A not on L. Suppose $\overline{BC}$ does not meet L. Will L meet $\overline{AB}$ or $\overline{AC}$ or both?
5. Given line L and point A not on L. Suppose $\overline{BC}$ meets L. Will L meet $\overline{AB}$ or $\overline{AC}$ or both?
6. Suppose line L meets segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$. Will L meet $\overline{AE}$?
7. Make up some problems similar to Problem 6 and solve them.
8. Construct a circle. Construct several.
9. Given point A, construct a circle with center A. Construct several if you can.
10. Given points A and B. Construct a circle with center A that contains B. Construct several if you can.
11. Construct a circle with given center and given radius. Construct several if you can.
12. Given a circle, construct a circle that lies inside the given one. Construct another. How are the two constructed circles related. For example, do they have the same center? Does one lie inside the other?
13. Given a circle, construct a circle that lies outside the given one. Construct another. How are the constructed circles related?
14. Given a circle with center A. Mark a point B inside the circle. Construct segment $\overline{AC}$ congruent to $\overline{AB}$. Where does C lie? Test several such points C. The same if $\overline{AC}$ is smaller than $\overline{AB}$. If $\overline{AC}$ is larger than $\overline{AB}$.
15. Given a circle with center A. Mark a point B outside the circle. Construct segment $\overline{AC}$ congruent to AB. Where does C lie? Test several such points C. The same if AC is smaller than $\overline{AB}$. If $\overline{AC}$ is larger than $\overline{AB}$.
16. Given a circle, mark point A and point B inside it. Construct AB. What do you observe? Try several cases.

17. Given a circle, mark point A and point B outside it. Construct $\overline{AB}$. What do you observe? Try several cases.

18. Can you find other interesting questions involving a circle and two points?

19. Given a circle, mark points A, B, and C inside the circle. Construct $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ to form a triangle. Mark point D inside the triangle. How is D related to the circle?

20. Given a circle, mark point A inside it. Construct $\overline{BC}$ containing point A. How is BC related to the circle? Try several cases.

21. Given a circle with point A marked inside it. Construct line L containing A. How is L related to the circle? Try several cases.

22. Given a circle with point A marked on it. Construct line L containing A. How is L related to the circle. Try several cases.

23. Given line L, construct a circle that doesn't meet L.

24. Given line L, construct a circle that meets L in two points.

25. Given line L, construct a circle that meets L in just one point. (this is hard to do at this stage but will be easier later. Note that we are trying to construct a circle, not just to find one by trial. However, the student should not be discouraged from applying trial and error methods.)

26. Given a circle, construct a line that meets the circle in two points.

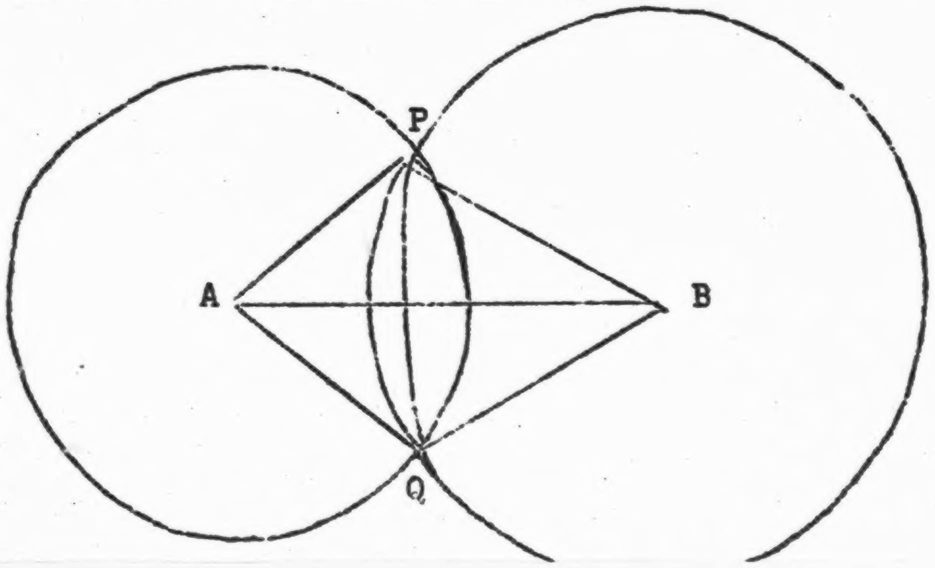
27. Given a circle, construct a line that meets the circle in just one point (Difficult).

28. Given a circle, construct a line that doesn't meet the circle.

29. Given a circle, construct a circle that doesn't meet it.

30. Given a circle, construct a circle that meets it in just one point. How is this point related to the centers?

31. Given a circle, construct a circle that meets it in two points. Note how the points are related to the centers of the circles. Let A and B be the centers; let P and Q be the points of intersection. How are $\overline{AP}$ and $\overline{AQ}$ related. How are $\overline{BP}$ and $\overline{BQ}$ related?



Sometimes we say A is equidistant from P and Q, similarly B is equidistant from P and Q. Construct line AB and line PQ. These are examples of perpendicular lines.

32. Given point A and point B. Mark a third point P. Construct the circle with center A that contains P. Construct the circle with center B that contains P. What do you observe about these circles? Try several cases. If the circles intersect in a point Q, different from P, how are P and Q related to line AB? How is line PQ related to line AB?

33. Given line L and point P not on L. Construct a line that passes through P and is perpendicular to L. Try to construct another.

34. What happens in Problem 33 if P lies on L?

Some activities that involve properties of angles and triangles are the following:

1. Given $\triangle$ ABC and segment $\overline{DE}$ congruent to $\overline{AB}$. Construct a triangle congruent to $\triangle$ ABC with $\overline{DE}$ as one side.

2. Given $\triangle$ ABC, construct a triangle congruent to $\triangle$ ABC with $\overline{AB}$ as one side.

3. In Problem 1, can you construct a second triangle that fits the conditions? Several?

4. In Problem 2, can you construct a second triangle that fits the conditions? Several?

5. Compare your answer to Problems 3 and 4.

6. Given points A and B, construct C such that $\overline{CA}$ is congruent to $\overline{CB}$. Can you find a second such point? Several?
   Definition: If $\overline{CA}$ is congruent to $\overline{CB}$ we call $\triangle$ ABC an isosceles triangle.

7. Given segment $\overline{AB}$. Construct an isosceles triangle $\triangle$ ABC such that $\overline{CA}$ is congruent to $\overline{CB}$. Construct another such triangle $\triangle$ ABC. Mark the intersection of line $\overline{AB}$ and line CD as E. Then $\overline{EA}$ is congruent to $\overline{EB}$ and E is called the midpoint of $\overline{AB}$.

8. How many pairs of congruent triangles can you find in the figure of Problem 7? Anything else of interest?

9. In Problem 7, choose ABC so that $\overline{DA}$ is congruent to $\overline{CA}$. This makes the construction shorter.

10. Given $\overline{AB}$, construct its midpoint and label it C. Construct D, the midpoint of $\overline{AC}$. Construct E, the midpoint of $\overline{AD}$.

11. Given segment $\overline{AB}$. Construct its midpoint C. Construct a point D such that $\overline{DA}$ is congruent to $\overline{DC}$. Draw line $\overline{CD}$. Then lines $\overline{CD}$ and $\overline{AB}$ are an example of perpendicular lines. (See Section C, Problem 31).

12. Given line L and point P on L. Construct a line that passes through P and is perpendicular to L. (Now we are better prepared to tackle Section C, Problem 27.)

13. Given line L and point A on L. Construct a circle which meets L just at A. Construct several such circles. How are they related? How are their centers related? Describe the figure formed by the line and the circles.

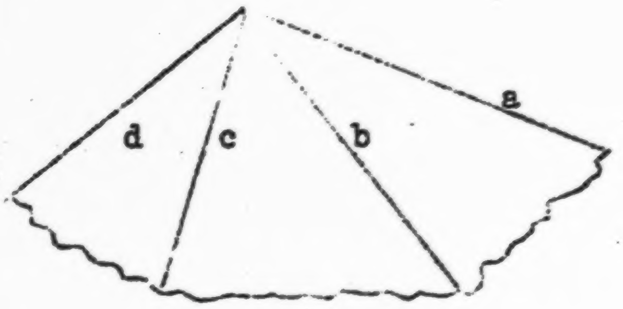
14. Given circle C and point A on C. Construct a line which meets C just at A. Construct several such lines. How are they related? Describe the figure formed by the circle and the lines.

15. Given line L and points A and B not on L. Try to discover whether the circle with center A that passes through B meets L, but do not draw this circle. Discuss various cases.

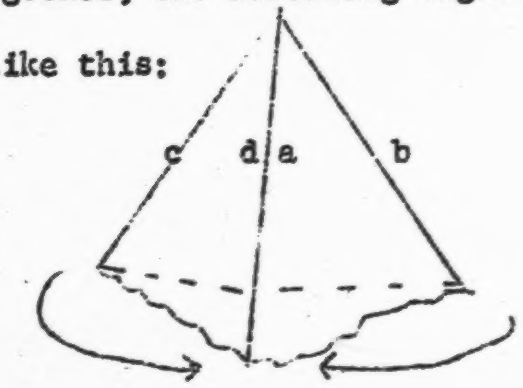
These suggestions are not intended to be exhaustive. Many books and pamphlets on straight edge and compass constructions are available for reference. What is important here is that motivation be provided for these activities and that a spirit of inquiry be inherent in the performance of those tasks.

CONSTRUCTING POLYHEDRAL ANGLES

Consider the following figure:



If this figure were cut out, folded along the lines b and c so that line a falls on line d, and then taped together, the resulting figure would be a trihedral angle that looks something like this:



Some questions that might be raised are the following:

(1) Is it always possible to fold a pattern like the first figure into a trihedral angle?

(2) Given fixed angles between b and c, and between c and d, what is the smallest angle that a and b can make so that the resulting figures can still be folded into a trihedral angle?

(3) What is the largest angle a and b can make so that the resulting figure can be folded into a trihedral angle?

These questions can be answered even before angle measure is discussed by having the students draw pictures of the angles that serve as greatest lower and least upper bounds. Of course, once the notion of degree measure is discussed, the above questions can be answered in terms of inequalities. At all stages, the answers given by the students can, and probably should, be verified by construction.

This work extends nicely to polyhedral angles with more than three face angles by adding one face at a time. For example, given three face angles, the greatest lower bound for the fourth face angle will be zero if the given three face angles can form a trihedral angle and will be the difference of the two smaller from the greatest of the three given angles.

Section IV - Chapter 10

GRAPHS, TREES & CONVEX POLYGONS

At this stage the students should have become familiar with geometric objects (one, two and three dimensional) which exist in the classroom. They should have clear notions of "straight line", "flat surface", "curved surface", "corner" etc. The objective of this section is to give the student experience in constructing geometric objects and noticing some of their properties.

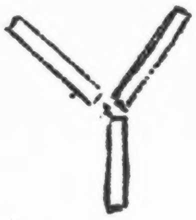
Construction of geometric objects by paper folding and the drawing of geometric figures on paper are considered in other sections. Here we use sticks for construction one and two dimensional objects and cut three dimensional objects from potatoes.

I. GRAPHS

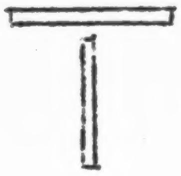
For these constructions we use sticks of various lengths. No particular relationship among the lengths of the sticks is desired; the properties we consider here don't depend on the stick lengths.

1. Trees

Sticks are laid on a table so that if any two sticks touch, then they touch at their ends. Thus we allow



but not



Two more conditions are required for a tree. The aggregation of sticks must be connected and must contain no loops.

These conditions should not be mentioned explicitly but should come out by analogy with real trees.

Construct lots of trees and non-trees until the students can easily decide when the sticks form a tree.

Call the sticks edges and points where one or more stick ends are present vertices. By counting lots of cases, become convinced that:

For any tree there is always exactly one more vertex than there are edges.

$$V - E = 1$$

The students can probably get an idea of a proof of this fact by asking how the numbers of edges and vertices are changed when an edge is added to a tree to form a new tree.

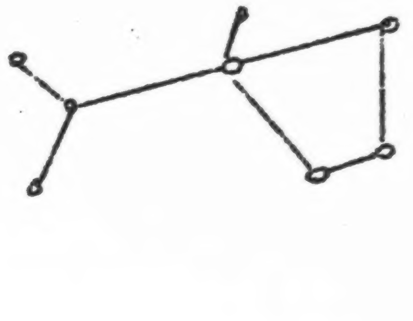
2. General graphs

Sticks are put down as before (so that if two touch at all they touch at their ends), but we no longer require connectedness or absence of loops. Thus all trees and non-trees considered above are allowed and these figures are called graphs.

Define the order of a vertex to be the number of edges (sticks) coming in to the vertex.

Practice giving orders of vertices on constructed graphs. We say that a vertex is odd if its order is an odd number, even if its order is an even number. Practice counting the number of even and odd vertices in graphs.

Example:



This graph has 3 vertices of order 1.
3 vertices of order 2.
1 vertex of order 3.
1 vertex of order 4.

Thus the graph has 4 odd vertices and 4 even vertices.

By examining lots of cases become convinced that:

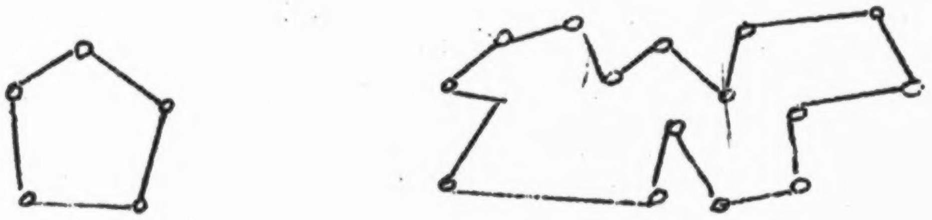
The number of odd vertices in any graph is an even number.

Again the students can probably see an inductive proof by asking what happens when a new edge is added.

3. Simple closed curves

An important special kind of graph is the simple closed curve, for which the criterion is easy: each vertex has order 2.

Examples:



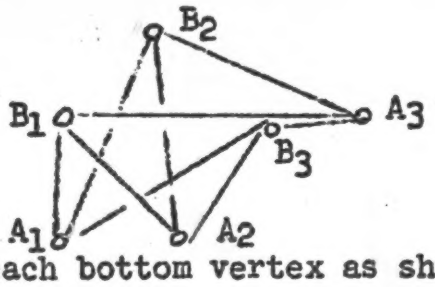
Notice that for a simple closed curve the number of vertices and edges are equal, V = E.

Notice that a simple closed curve divides the plane (table) into a part inside the curve and a part outside the curve.

4. Graphs in 3-space

For this we need to be able to attach the ends of sticks so that the graph will hold together. Tinker toys should do the job if the "vertices" allow "edges" to come out in enough directions.

Example: Put 3 vertices A1, A2, A3, in a plane (i.e.on the table) and 3 vertices B1, B2, B3, above.



Connect each top vertex to each bottom vertex as shown. (Possibly call bottom 3 houses and top 3 water, light, gas plants. Then each utility must be furnished to each house.)

See if one can make the same graph in a plane. Use wires which can be bent to form the edges. Become convinced that even then this graph cannot be put in the plane.

Conclude that: Not all graphs can be drawn in the plane.

II. POLYGONS

When we considered graphs we were considering one-dimensional objects. It is true that we made the graphs in a plane or in space, but the graphs themselves were made up of edges, which are one-dimensional In this section we consider two-dimensional objects.

1. Intersection of polygons

We noticed above that a simple closed curve (made of edges) divides the plane into an interier region and an exterior region. We call this interior region a polygon.

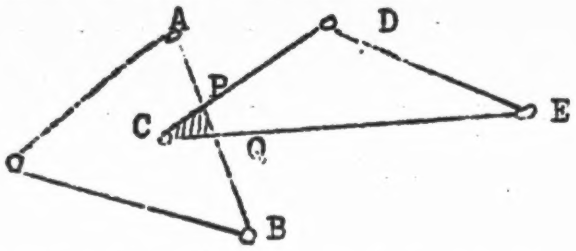
If the boundary simple closed curve of a polygon has 3 edges (exactly) the polygon is a triangle. If it has 4, the polygon is a quadrilateral.

If more, we just call it a polygon with ___ sides.

Construct some polygons. Learn to draw polygons on paper.

Consider polygons which may overlap. Look at the common part.

For example, the shaded area is the common part of the two triangles.



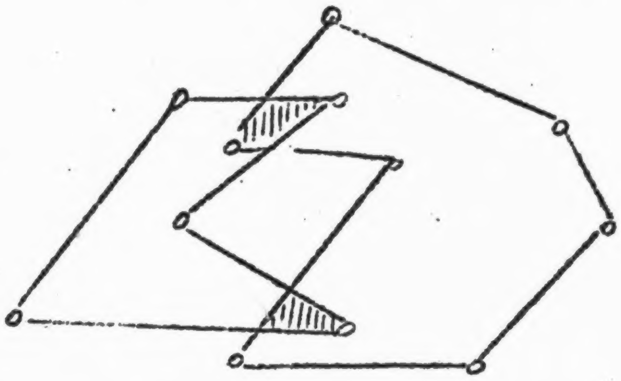
The two triangles together don't make a graph because edges touch (actually cross) at points which are not ends of the edges. But we can make this into a graph if we use more and shorter edges. Instead of the stick AB we use three sticks AP, PQ, QB. Similarly, we make CD into two sticks CP, PD and make CE into two sticks CQ, QE.

(It may be best to cut the original sticks into appropriate pieces for the first few examples. After that we can imagine cutting them and just draw the results on paper.)

Notice that for this example, once we have cut the sticks, the common part of the two triangles is itself a triangle.

Things don't always work this nicely when we consider the common part of two polygons.

Example:



Here (after cutting edges where they cross) the common part consists of a triangle and a quadrilateral.

Examine many intersections.

(i) Intersect triangles to get a quadrilateral.

(ii) Intersect quadrilaterals to get one triangle.

(iii) Intersect quadrilaterals to get two triangles.

(iv) Intersect two polygons to get a 5-sided polygon.

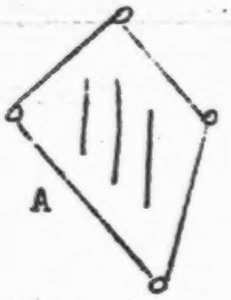
etc.

Become convinced that: The common part of two polygons is one or more polygons.

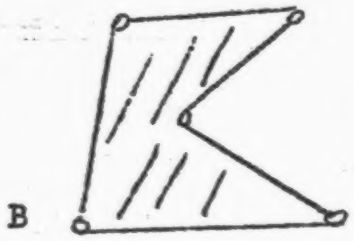
2. Convexity

We noticed that when we took the intersection (common part) of two polygons we sometimes got one polygon and other times we got more than one. We will now consider some special polygons which have the property that the intersection of any two of them is a single polygon and again one of the special kind.

Consider the difference between



and



Imagine two-dimensional creatures living in these polygons. In A any two such creatures could always see each other, but in B they may be "around the corner" from each other. Another way of saying this is that for polygon A if a straight stick has both ends in A then the stick must lie in A, but this is false for B.

Call a polygon convex if any straight stick with its ends in the polygon must lie in the polygon.

Draw lots of polygons. Decide which are convex and which are not convex. Consider intersections of convex polygons. Examine enough to become convinced that:

The intersection of two convex polygons is a convex polygon.

See if students can decide (by logic) if the intersection of three convex polygons will be a convex polygon.

Number vertices in order around polygons (starting point is not important). Call a line segment (or stick) which goes from a vertex n to vertex n + 2 (i.e. skips one) a testing line segment.

If a polygon is convex then all testing line segments are in the polygon (some may lie completely along the boundary--this is considered to be in the polygon). Become convinced that: If all test line segments lie in the polygon, then the polygon is convex.

III. SOLID FIGURES

The solid figures may be cut from potatoes. Probably the teacher will have to do most of the construction here, but the students can handle the models, count edges and faces, etc.

Construct cubes and more general parallelepipeds. Define face to be a plane polygon on the boundary. Edges are the boundaries of these faces (one edge will be part of the boundary of two faces). Vertices are points where edges come together.

Count number V of vertices
number E of edges
number F of faces

for many solid figures. (Point out that some solid figures have been given names, and use names.)

Make a table.

Figure	V	E	F	V - E + F
cube	8	12	6	2
tetrahedron	4	6	4	2
pyramid	5	8	5	2
etc.				

Finally become convinced that: for any solid figure $V - E + F = 2$.

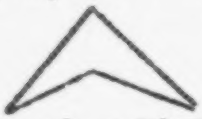
TESSELLATIONS

This material can be presented quite early, some in kindergarten and some in the first and second grades. The approach is primarily empirical, and the few mathematical arguments are quite simple.

I. Tessellations of a plane.

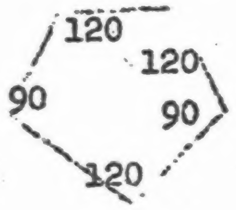
Give the student a bunch of congruent triangular blocks and ask him to start tiling the floor (or table) with them. Then use other triangular blocks, again all congruent. The student should become convinced that he can tile the plane with any kind of triangular blocks, provided all blocks are congruent. It should be noted that at any one vertex we have the three different angles of the triangle, each occurring twice. Also this gives a visual demonstration that the angles of any triangle add up to a straight angle.

Next we tile (or tessellate) the plane with quadrilateral blocks. It is easy to see how to do this with rectangualr blocks, but it is not obvious that it can be done with any quadrilateral. So many different quadrilaterals should be used so the student becomes convinced that it is always possible. For example, tessellate with blocks like



In this case, we find that at each vertex all four angles are present. This shows that the angle sum of any quadrilateral is 360°.

When we try to tessellate the plane with pentagons, the situation is quite different. The students should easily convince themselves that one cannot tessellate the plane with regular pentagons. Furthermore, they will probably see that this is because the angle sums don't add up right. If only three come into a vertex a gap is left, but four would overlap. However, we can tessellate with some pentagons. For example, blocks like



will tessellate the plane.

It may be interesting to let the students play with blocks of several shapes at once to form combinations which will tessellate. In some cases they should be able to discover that some combinations form a triangle or a quadrilateral, thus assuming the possibility of tessellations with such combinations. Also, tessellation with some other polygons (for example, regular hexagons) can be seen to be possible.

II. Other tessellations.

Sometimes we may want to tessellate a plane region with a boundary so that we "come out even" at the boundary. For example, if we want to tile the floor of a room we want no gaps at the walls in addition to wanting the tiles to fit together without gaps on other parts of the floor. This changes the problem drastically. For one thing, when tessellating the plane we didn't care about the size of the blocks, only about their shape. For another, if at some stage the edge of our array of blocks was of a strange shape, we didn't care because we planned to continue adding more blocks indefinitely (or at least to conceive of doing so).

One interesting illustration uses a checkerboard and rectangular blocks such that each block will cover exactly two squares of the checkerboard. It is easy to tessellate the board with such blocks. However, if we try to tessellate all of the board except two diagonally opposite corner squares with these blocks, we find this to be impossible. (Proof: Each block covers one white square and one black square, and we no longer have equal number of white and black squares.)

In grades four, five, and six the students might try tessellations of cylinders and spheres with curved blocks. It is interesting to notice that although the plane cannot be tessellated with equal regular pentagons, the sphere can be with twelve equal regular (curved) pentagons. It is probably best to stay away from surfaces (such as the torus) whose curvature is different at different points because a curved block which will fit one place will fail to fit at some other places.

DISSECTION OF FIGURES

Figures can be classified into physical objects and drawings, for present purposes. They also fall into *dimensional classes*, i.e., 3D-in-3D, 1D-in-2D, etc. Each of these is subdivided into *types*; the 3D-in-3D into balls, bricks, solid polyhedra, polygons, simple closed curves, etc., as examples.

A *dissection of a figure* is really a *partitioning*, by physical cutting, or by lines on a drawing or on a physical object, of the figure into subfigures that are (essentially) pairwise disjoint and have the original figure for their union. Dissections can be classified by the types of figures that occur in each partition, the number of each type, and relations between the figures in the partition: are all the triangles congruent? Similar? etc.

The remainder of discussion shall consider only partitions into figures of the same *dimension class*, for sake of time, effort, and sanity. Thus we shall not consider such problems as partitioning Mariner IV into line segments, spheres, bricks, and bent pipe cleaners.

DISSECTION OF PHYSICAL OBJECTS
(or KITCHEN GEOMETRY CONTINUED)

Examples of physical objects: potatoes, bananas, doughnuts, wedges (cake slices), cones, solid cylinders (brown bread, cranberry jelly), bricks (ice cream, wooden plans, etc.). Oranges. Jello. Balls. Pipes

All paper materials: the works. Straws, cartons, rolls, shirt cardboards, drinking cups,...

Plastics. Thin metal lamina. Rubber balloons.

Thread, string, wire, rope, plastic. Spider web. Rope coils.

Soap Films on wire frames.

I. One object

Since this is dissection, we start with a particular object and ask for its dissection(s). First, of course, must develop idea. Given figure and a dissection, is it a dissection of the figure. Asking for all dissections is too much. But we may ask for the following, for a given figure F:

(i) Find some dissection. Find several.

(ii) Find a dissection that has two figures; e figures; etc.

(iii) Find a dissection of F into two congruent figures; two similar figures; n congruent; n similar.

(iv) Dissect F into figures all of which are of a specified type; of specified types and numbers.

(v) Dissect F into figures all of which are congruent (similar).

Variety of others, obtained by varying the characteristics of the dissection (see introductory paragraph above), and depending on original figure, its shape and material.

A. Object is 3D-in-3D. Partitioned into

(i) Finitely or infinitely many 3D-in-3D figures. This includes 3D-in-3D tesselations. For problems, see (i) - (v) + above.

(ii) Infinitely many (finitely many impossible) 2D-in-2D, i.e., cutting into "slices". Now slice ball? Doughnut? Brick? Cylinder? Must all slices be parallel? Comparison of slices from different slicings; e.g., doughnut yields discs one way, annular another, mixture another.

(iii) Infinitely many 2D-in-2D, that is, curls, as with wood. Ball into spheres, solid cylinder into cylinders (open-ended or not), solid wedge into roofs, etc.

(iv) Infinitely many 1D-in-1D, i.e., line segments.

(v) Infinitely many 1D-in-2D, such as triangles, circles, quadrilaterals, etc., What can be "fibered" into what?

(vi) Infinitely many 1D-in-3D. Bent wires, twisted circles. This tricky.

B. Object is 2D-in-3D. Rich class: surfaces of balls, bricks, doughnuts. Open-top boxes (cardboard cartons). Partition into

(i) 2D-in-3D. Tesselation of sphere, torus, etc.

(ii) 2D-in-2D. Only cases are either trivial or impossible.

(iii) 1D-in-3D. Crazy tesselation.

(iv) 1D-in-2D. Not so wild. HERE GET CONICS.

(v) 1D-in-1D. Disciplined. Ruled surfaces.

C. Object is 2D-in-2D. Partition into

(i) 2D-in-2D. Paper folding. All plane tesselations: here can play this meta-game in concocting problems, kits, and games: three inputs, namely, figure to be tiled, permissible tiles, and tiling rules, and one output, a tiling, Ring the changes. Also, recall (i) - (v) + of introduction, including dissection into 2 congruent figures: role of symmetry (lack of it puzzle basis).

(ii) 1D-in-2D. Partition disc, square, etc., into bent pins, triangles, circles, etc.. Or have many (infinitely many?) kinds in one dissection.

(iii) 1D-in-1D. Cutting plane regions into line segments, not necessarily all parallel.

D. Object is 1D-in-3D. Partition into

  (i) 1D-in-3D

  (ii) 1D-in-2D

  (iii) 1D-in-1D

E. Object is 1D-in-2D. Partition into

  (i) 1D-in-2D

  (ii) 1D-in-1D

F. Object is 1D-in-1D. Partition into

  (i) 1D-in-1D. Folding paper straight edge; wire; string.

  (ii) 0D-in-1D. Folding paper straight edge.

Need material on two decompositions of one figure.

II. Two physical objects.

Having worked on dissections of single figures, can now work on problems such as the following:

(i) Given a figure and a dissection of it, find another figure that has the same dissection. (This is really a composition problem.)

(ii) Given a figure, find a dissection and another figure that has same dissection.

(iii) Given 2 figures, and a dissection of one: see if other figure has same dissection.

(iv) Given 2 figures, try to find a common dissection.

(v) Given a figure and a dissection: see if latter is a dissection of former.

DISSECTION OF DRAWINGS

For bounded drawings, discussion parallels that of physical objects, except may have richer zoo to work with.

Essentially new element here is class of unbounded drawings. Here can occur separation theorems, if we wish. Line partitions plane, plane partitions space, point partitions line.

Finite set of lines partitions plane (2-color problem).
Finite set of planes partitions space (?-color problem).
Finite set of points partitions line (2-color problem).
Circle partitions plane: s. cl. curve does.
Parabola partitions plane.

What else? Plane into squares. Infinite tiling: Ulam's biology.

Need more work.

Section IV - Chapter 13

ORDER

This unit is designed to provide some experience--both intuitive and concrete--of gross comparisons of size for geometric objects of various types. While learning about the order relation, the child should learn to distinguish and identify many of the standard geometric figures. Among other things, this work (or rather play) is to serve as preparation for the later study of measurement.

There are no pre-requisites, not even knowledge of the integers beyond counting. It appears to us that this entire unit can be completed in kindergarten or first grade. The teacher should have no difficulty in formulating classroom games out of which the stated results will appear.

1. Compare two line segments (at first, sticks should probably be used instead of line segments). Introduce the standard notation for bigger than, smaller than, and equality. As a general principle, whenever the children are asked to make a comparison, they should guess at the results in advance.

2. Given a line segment, produce a bigger one and a smaller one. (It may be easier to work with string here.) Illustrate and discuss transitivity. As an example of a non-transitive ordering, one might use the "paper, scissors, stone" game.

3. Take 4 or 5 line segments and compare all possible pairs. Then arrange the segments in a table of ascending size and practice reading off comparisons from the table. (If the children are unable to read or write, the sticks may be identified according to color. Since color blindness is not infrequent, it would be helpful to have the sticks identifyable by their cross-sections.)

4. Given two line segments, produce one of intermediate sine. Repeat this process several times. Discuss how many times this can be done.

5. Combine (that is, add) a line segment with itself. Do this several times to make a long segment. Hint at the archimedean property. Introduce + as notation for combining.

6. Combine arbitrary line segments; observe commutativity and the associative law.

7. Consider 4 or 5 sticks (that is, line segments) of different sizes and colors; there should be several copies of each one. Form all possible sums of pairs. Arrange these sums in ascending order and practice reading from the table. Choose a pair and have the children ask questions about its order properties to decide which pair was chosen. How many questions are needed? Discuss adding inequalities for example, $a < b$ and $c < d$ imply $a + c < b + d$. Define some sets based on the order relation.

8 Verify that the sum of any two sides of a triangle is greater than the third side. (For a given triangle, this should involve writing out 3 or maybe even 6 inequalities.) Given three sticks, under what conditions do they form a triangle? Is it unique when it exists? What happens with 4 sticks?

9. Compare areas of similar planer figures: squares, rectangles, triangles, circles, quadrilaterals, pentagons, hexagons, wiggly figures. Give them names, as feasible.

10. Do #9 for non-similar figures. At this stage, placing one area (a flat block) on another should make the answer obvious. Make tables.

11. Combine areas to cover other areas--for example, the set of area blocks should, at least include all the faces of the ESI multidimensional blocks.

12. Cut up an area (paper) in order to compare it with a given area. Areas enclosed by polygons should be treated before circle. Do simplified versions of 3 and 7.

13. Give two areas for which one cannot decide which is larger.

14. Do volumes in analogous ways. Comparisons should be based on the use of a balance scale or the use of hollow objects which can be filled with water or sand or both. Identify cubes, boxes parallelapipeds, spheres, cones, tetrahedra, cylinders, etc., as feasible. Do lumpy things, too.

15. Illustrate some volume relations: for example, the volume of a cylinder is three times the volume of a cone with same base and height. The volume of these solids depends only on the heights when the base is fixed.

16. Treat lengths of curves in the plane in 3-space, and surface areas. The important thing here is forthe children to decide that unravelling, unfolding, or cutting up is the way to proceed. It shouldn't be hard here to make examples in which visual perception is wrong. Surface areas are troublesome and should, perhaps, be omitted.

17. Treat angles as in #1-7. It is not necessary, at this stage to define the notion of angle. The children should become familiar with straight and right angles; connect with paper folding.

18. Tear a paper triangle and at one vertex adjoin the other angles to get a straight angle.

3

Chapter 14

MEASUREMENT

Objectives. The role of measurement in our society is of fundamental importance. The development of the basic notions of measurement deserves considerable attention throughout all levels of experience in grades K - 6.

Among the basic notions which should receive attention are ordering a set of objects with respect to a given relative measure (such as length or weight), comparing relative measures with simple unscaled devices (such as string), making use of arbitrary units to assign measures to objects, and making use of standard units of measure to assign length, area, and volume measures to objects. It may be quite appropriate to develop some of these notions in conjunction with the development of the number line.

Students should handle the objects to be measured and compared, and should use the measuring devices and record their own data. Emphasis should be placed on the fact that the physical act of measurement yields approximations and that much depends on the tools at hand.

A. Ordering with respect to a given relative measure

1. Let students place sticks in order of length from shortest to longest. They should be able to do this by pairwise comparisons. No sort of measuring instrument should be used to perform this task. One property to note here is that of transitivity--if stick A is longer than stick B and stick B is longer than stick C, then stick A is longer than stick C.
2. Have students arrange themselves in order with respect to height. This is essentially the same sort of exercise as #1. It would probably be best to compare "shoes on" heights here. In cases where the children have trouble deciding whether or not Student A is taller

than Student B, let the students try to devise some method for making a decision. If no decision can be made regarding some pair of students, then agree that, for all practical purposes, that pair has the same height. The property of transitivity of the relation "is taller than" should again be noted. It may also be worth noting that if Student A is taller than Student B, then A is taller than each student who is as tall as student B.

3. Have students arrange themselves in order with respect to age. The likelihood of two students having the same birthday increases greatly, of course, as the size of the class increases. In the event that two (or more) students have the same birthday, agree that these students are of the same age.

   One can now ask the students to compare the list obtained for "is taller than" with that obtained for "is older than" in order to answer the following questions:

   Is it true that the older of two students is taller?
   Is it true that two students of the same age (height) are also of the same height (age)?

4. Determine the order of distances that students' homes are from school. This can be done by making use of scale maps of the school neighborhood that are "usually" drawn on this level.

   The students can now be asked the following questions:

   Is it true that the older of two students lives farther from the school?

   Is it true that the taller of two students lives farther from the school?

5. Place stones (or other suitable common objects) in order of weight from heaviest to lightest. Students should use a simple beam balance as an aid in making decisions.

6. Ask students to consider the list they obtained when they arranged themselves according to height.

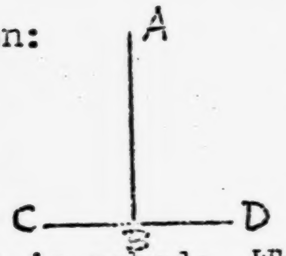
Have them consider questions like this:

> Student A is taller than Student B, and Student C is taller than Student D. If we combined the heights of A and C and of B and D, which combined height would be greater?
>
> Student A is taller than Student B. If we combined the height of A with any Student C and of B with the same Student C, which combined height would be the greater?

The combining of heights can be thought of as "child-stacking" or something similar to develop an appropriate story line. Looking at the results of combining heights leads nicely into the notions that if $a < b$ then $a + b < b + c$ and if $a < b$ and $c < d$ then $a + c < b + d$. It should be possible to even get at the notion that if $0 < a < b$ and $0 < c < d$ then $ac < bd$ by considering the relative sizes of rectangles of dimensions a by c and b by d.

B. Using unscaled devices (string, string compass, or the edge of an index card) to compare measures.

1. Have students use a piece of string or an index card to compare the lengths of segments drawn in various positions on a piece of paper or on the chalkboard. Some segments of "practically" equal lengths should be drawn on the same practice sheets. (It should be noted that vertical segments sometimes "seem" longer than horizontal segments of the same length. This is the source of the reasonably well-known optical illusion:



in which the question is asked: Which is longer, $\overline{AB}$ or $\overline{CD}$? (The answer, in this case, is that neither is longer, for they are the same length.) The need for a device such as a string as an aid in comparing lengths can be brought out quite nicely by having the students try to compare the lengths of various segments where visual comparison can be incorrect and where it is impossible to move the segments to be compared.

2. Use can be made of a string compass or index card to draw a segment which is as long as two given segments laid end to end. This will get at the notion of additivity of lengths. This type of activity will lay the foundation for development of the triangle inequality and for the notion that "it is as far from A to B as it is from B to A". This activity will also help to develop a feeling for the preservation of length under a rigid motion.
3. Practice adding lengths by making use of five sticks of lengths 1, 2, 4, 8, and 16 respectively, to obtain all integral lengths from 1 to 31. It is worthwhile to note the uniqueness of the combination of sticks needed to produce any one of these lengths from the given sticks. It is also worth noting that once a particular combination of sticks is chosen, the same length is produced regardless of the order of addition.

   If it is feasible to talk about the difference of lengths, practice in adding and "differencing" lengths can be given by making use of four sticks of lengths 1, 3, 9, 27, respectively, to obtain all integral lengths from 1 to 40. Uniqueness of representation and order of operations is also worthy of discussion here.
4. Have each student try to guess the location of the midpoint of a segment about 6 inches long. Each student should have a copy of the segment, and should do this without the aid of measuring instruments. Then have the students fold their paper (or measure) in order to locate the midpoint of the segment and decide whether their guess was to the right or left of the "true" midpoint. Tabulate the results and discuss whether the class thinks that they tend to guess towards

the left or the right.

5. Repeat 4, trying this time to guess the location of the "left hand" trisection point. After the students use measuring instruments to locate the left point of trisection, discuss whether the class tends to guess towards the right or left in this case.

C. Measurement with a non-standard arbitrary unit

1. Have each student measure the width of his desk using the span of his hand (the distance between the tips of his little finger and thumb). Then have each student measure the width of another student's desk in the same way. Discuss the following questions:

   Given that Student A measures two desks and finds that each is between 7 and 8 spans, can (or, should) he conclude that both desks are the same width?

   Student A finds his desk to be 8 spans wide and Student B finds his desk to be 10 spans wide. Can we conclude that B's desk is wider than A's?

   Given that Students A and B measure the same desk and A finds the desk to be 8 spans wide while B finds the desk to be 10 spans wide, can we conclude that either A or B counted incorrectly? Can we conclude that the desk is both 8 spans and 10 spans wide?

   Problems of this sort should serve to illustrate the need for the establishment of some fixed "standard" measures.

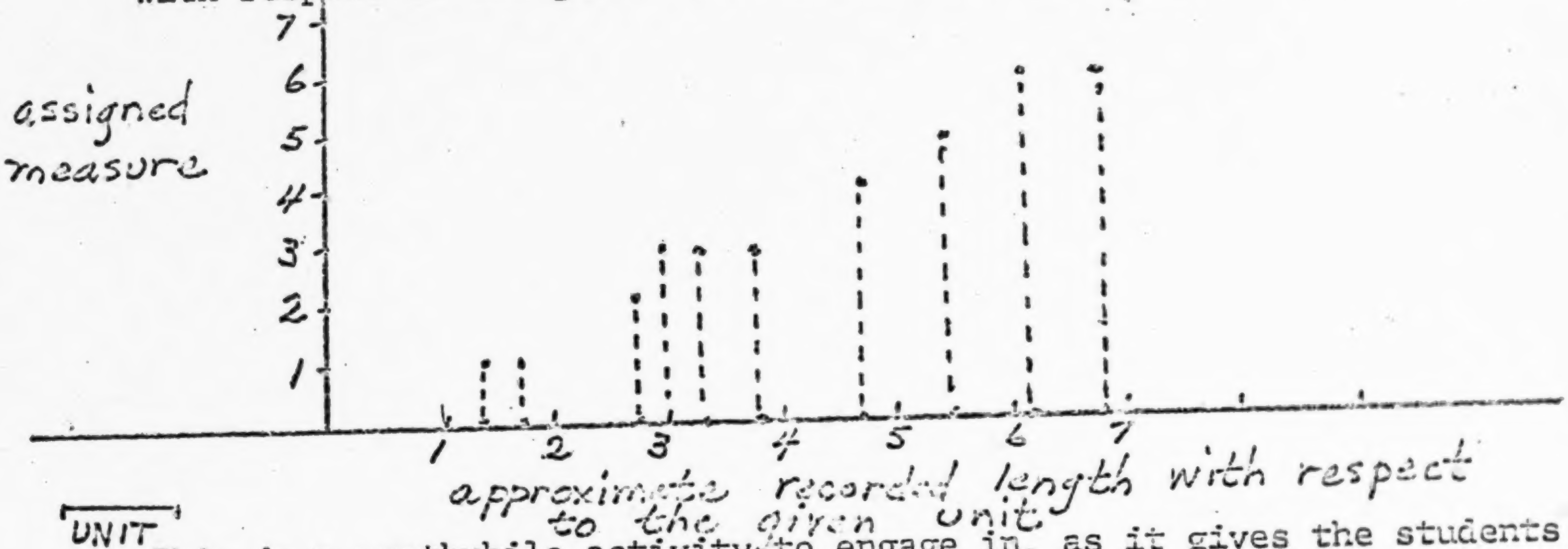
2. Have students use a "small" fixed length to measure the sticks that they had previously ordered according to relative lengths. This will give them classes of sticks in various ranges, such as those between 3 and 4--that is, at least 3 but less than 4--of our fixed unit, between 4 and 5 units, etc.

   If we now agree to say that all sticks which are between 3 and 4 units have measure 3 of our "standard" units, then one thing worth noting here is that no two sticks of measure 3 in "standard" units differ from each other by more than 1 unit.

It may be worthwhile here to note the answers to the questions asked in C1 if the students measure their desks with a given "standard" length.

It may also be worth noting that transitivity of "being shorter than" still holds since any stick of, say, measure 3"standard" units is shorter than stick of 4 "standard" units, and any stick of 4 "standard" units is shorter than any stick of 5 (or more) "standard" units.

One way to record the data on the measures assigned to the sticks with respect to the given unit is illustrated below:



This is a worthwhile activity to engage in, as it gives the students practice in making two-dimensional graphs and illustrates a reasonably efficient procedure for recording and organizing data. Graph can be made on chalk board or on a felt board.

Changing the unit of measure to one which is, say, half as long as the initial one and making a graph of the "new" measures assigned to the same sticks will emphasize both the similarities in appearance of the graphs of this type as well as the effects of changing the size of the unit of measure.

3. Have students use beam balance and washers (of the same size) to assign weight measures to the rocks which were previously ordered according to their relative weights. Make a graph of the results.

Get washers each of which is about half the weight of an original one and assign "new" weight measures to the rocks with respect to this new unit of measure. See if the students can guess the number of smaller washers needed. Make a graph of the results and compare with the earlier graph.

4. Discuss what is meant by measuring to the nearest unit. The students will have to have a feeling for "half of", "less than half", and "more than half" in order to understand this notion. Given this understanding, have the students measure their bunch of sticks to the nearest of some prescribed unit. If this is done with the same unit that was used in exercise 2, then the graphs could be compared for similarities and differences.

5. With the concept of measuring to the nearest unit developed to some extent with linear measure, discuss the possibilities of weighing the rocks (exercise C3) to the nearest unit. The need for a "half washer" will probably arise here.

6. Draw a reasonably large triangle. Determine the location of the midpoints of the sides of the triangle (perhaps by folding). The midpoints of the sides of the original triangle determine a second triangle. Compare each side of this new triangle with the sides of the original triangle. Compare the perimeters of the triangles. Compare the areas of the triangles (perhaps by tiling the larger with copies of the smaller one).

7. The technique for treating volume measure should be based on the use of hollow figures which can be filled with water or sand, or by using a balance with solids of constant density.

We assume the existence of a large collection of 3-dimensional objects (with many copies of each one) suitable for balance-weighing or for pouring. Each child is to select for himself (or construct) an arbitrary unit of volume. Since it is not easy to reproduce this volume precisely, pouring is the preferred technique here. The volumes of given objects should then be measured--the results to be stated usually as between two consecutive integers, and possibly a choice of the nearest integer should be made. Scaling might be illustrated by multiplying dimensions of a box by small integers. Figures can be combined and the corresponding inequalities added (if only pouring is used, it would be nice to be able to remove faces). The unit volume might be halved by trial and error and this new unit used for closer approximations. The child should become convinced that his unit of volume can serve to measure volumes of solid figures, but may not give the same measures as someone else's unit. The teacher can pursue this line as far as taste and desire require--the finale being the need for a common unit.

8. Choose a particular cube as the common unit of volume. (Such a choice, the question of scaling, etc. underlie our preference for doing volumes after length rather than before it.) With this unit of volume, the things done in #7 can be repeated. The students should compare their answers and, as usual, whenever a volume is considered it should be estimated in advance. By repeated work with rectangular boxes of integral dimensions, the students should discover the formula for their volume. Discuss the principles of arithmetic that can be illustrated by combining volumes.

   Get ½ the unit volume by halving one dimension (from work on line segments the children can do this). Halve all three dimensions of the unit cube to get a cube whose volume is 1/8 of the original cube. Given an arbitrary box, practice refinements and approximations to its volume.

Measure the volume of standard figures by refining approximations. Among the standard figures to consider are the sphere, circular cylinder, circular cone, parallelepiped, and prism. Among the relations that can be noted are that a cone with base area B and height h has one third the volume of a cylinder with the same dimensions, and that a sphere whose diameter is d has a volume which is two-thirds the volume of a circular cylinder with diameter d and height h.

9. Choose a basic square as the unit of area. Compute areas of squares and rectangles with integral sides by counting boxes. Construct some figures with area specified in advance. Find the area of a right triangle whose shorter sides have integral length. Decompose parallelograms into rectangles with same height and base. How could one find the area of any triangle?

   Work with rectangles whose sides are allowed to be of form integer $+\frac{1}{2}$. Find areas by counting halves and quarters of the unit square. Move on to rectangles with rational sides--operationally, these are the only ones that occur for the children.

10. Compute inner and outer approximations to the area of a simple closed curve by using a basic unit square. Refine these approximations using a square whose dimensions are halved. Discuss what would happen as smaller squares are used. Compute the area of a circle whose radius is twice as long as the side of the basic unit square.

D. Measurement with standard units of measure

1. Introduce 1 inch, 1 foot, and 1 yard as standard units of linear measure. Students should get measures of "reasonable" distances and lengths in the classroom to the nearest yard, nearest foot, and nearest inch. Questions of this sort might be asked:

If two desk tops are each found to be 32 inches long, to the nearest inch, then are the desks necessarily the same length? If not, what can be the greatest difference between their lengths?

If two rooms are each found to be 10 yards long, to the nearest yard, then what can be the greatest difference between their lengths. How many feet is this difference? How many inches is this difference?

2. Assuming that there has been some discussion of the meaning of area of planar regions, it would be appropriate to have the students measure the lengths and widths of various rectangular shapes to, say, the nearest inch and then to give estimates of the areas of these shapes in square inches. Try to decide which of the completed areas are overestimates and which are underestimates of the "true" area. Next, have the students measure the lengths and widths of the same rectangles to the nearest half inch. Discuss the fact that a square 1 inch on a side can be divided into four squares each ½ inch on a side. So, in effect, 1 sq. in. = 4 sq. (½ in.).

Have the students give estimates of the areas of the rectangular regions in square inches, using the nearest ½ inch measures. Try to decide which of these computed areas are overestimates and which are underestimates. Also discuss whether these estimates of area are "better"--that is, closer to the real areas--than the first estimates.

The arithmetic involved here can get a little messy, but getting good estimates of the areas in question is a worthwhile activity. Making tables of values ought to halp in keeping the details straight. Here is one kind of table that could prove to be useful:

Rectangle 1

	Dimensions	Area
nearest inch	4 in. x 3 in.	12 sq. in.
nearest ½ inch	9(½in) x 6(½in.) or 4½ in. x 3 in.	54 sq. ½-in. or 13½ sq. in.
nearest ¼ inch	18(¼in.) x 12(¼in.) or 4½ in. x 3 in.	216 sq. ¼-in. or 13½ sq. in.

etc.

3. Have three or four randomly spaced points marked on a segment about 7 inches long. Students should measure the lengths of the consecutive "small"parts and the length of the whole segment to, say, the nearest ½ inch. Check the sum of the measures of the parts against the measure of the whole segment. Repeat the measuring process to the nearest ¼ inch,nearest 1/8 inch, etc.

4. Standard volume measures and weight measures need to be introduced in such a way that the students are as actively engaged in measuring volumes and weights of common objects or containers to the nearest specified fraction of a standard unit as they were when using their arbitrary units of volume and weight. These activities should strengthen and reinforce the concept that the measures derived are, at best, reasonable approximations to the "true" measures.

S. Szabo - 8/5/65 Re-write

based on

"Measurement", by E. Weiss
" , by S. Szabo

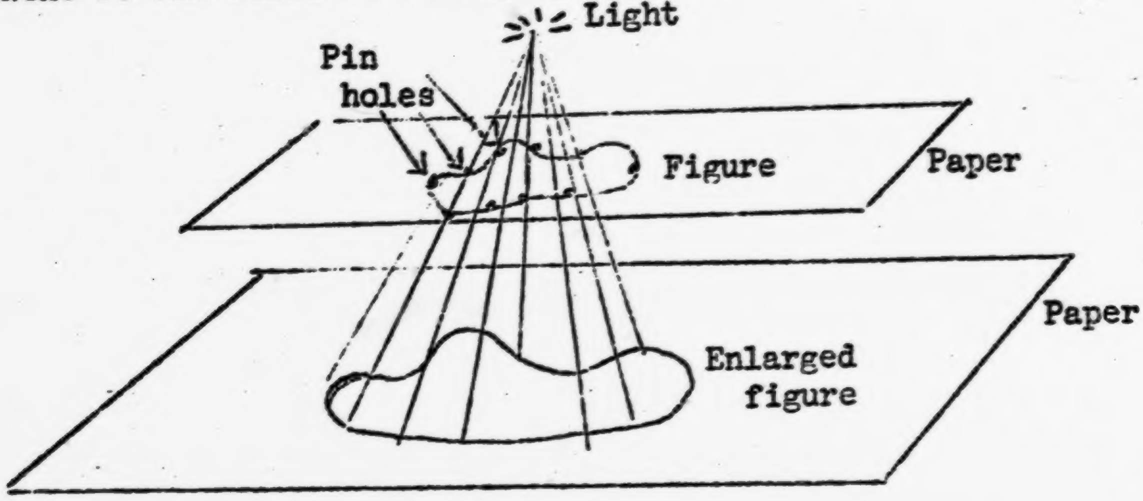
SIMILARITY AND MAP MAKING

We divide map making into two parts. The first concerns making maps of regions which are sufficiently small to be considered planar. In this case it is simply a question of changing the scale, that is, it involves the notion of similarity. This part can probably be done in the second or third grade. The other part concerns transferring a map from a spherical surface to a plane by various methods. This is much more complicated, but it could probably be done in the fifth or sixth grades.

I. Similarity and maps of plane regions.

The concept of similarity of geometric figures can be introduced by constructing triangles, quadrilaterals, etc., with sticks. For example, have a set of blue sticks twice as long as some red sticks. Then graduate to sticks of lengths, say, 1, 2, 3, 4, and make figures with various combinations. It is important for the student to notice that two triangles will be similar if their sides are proportional, but that this is false for polygons with more than three sides. Discuss how many angles of a polygon must be checked to make sure of similarity (when sides are already known to be proportional).

Next we can build an enlarging machine.

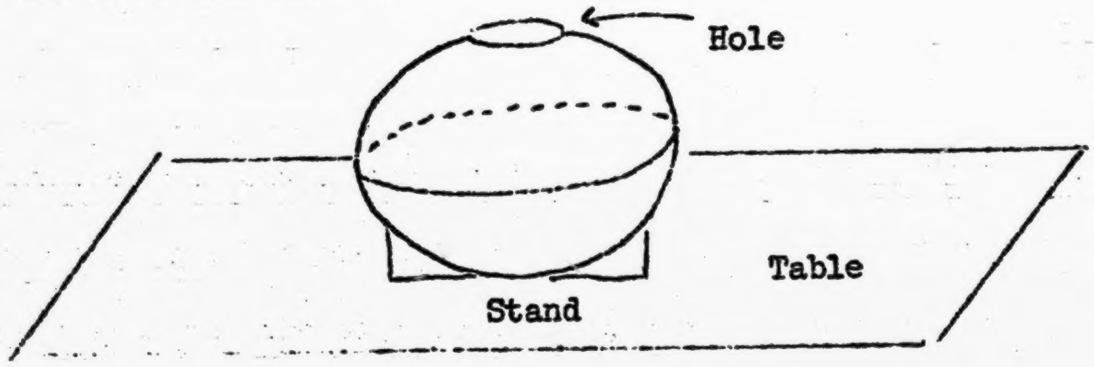


Notice that the enlarged figure will be similar to the original if the papers are parallel, but there will be distortion if they are not. It may be worth studying the different amounts of enlargement for different areas when the two papers are not parallels.

Now we can explain the process of map making as a similarity transformation. Make scale maps of the room, the building, a city block, etc.

II. Projecting from spheres.

For this we need a large, hollow, transparent plastic sphere, say about 2 feet in diameter. It should have about a 4-inch hole at the top and a stand at the base.



First we discuss latitude and longitude and draw various meridians and latitudes. A discussion of time zones and change of days would be appropriate and helpful. With colored grease pencils, draw various maps and finally a globe.

Pose the problem of transferring a map on the sphere to a flat piece of paper. Let the students try putting translucent paper on the sphere and try to trace the map. Let them become convinced that this method is not good because the paper cannot be made to fit.

Introduce a small penlight with a sharply-focused beam. See if the students will be able to suggest all of the following projections.

(i) Central projections.

Put paper on the table. Extend your arm with flashlight to the center of the sphere and shine the light toward the paper to transfer a map in the southern hemisphere to the paper. Look at the images of various meridians and latitudes. Discuss the amount of distortion. Where is there the least distortion and where the most? What curves on the spheres become straight lines on the paper? What are the images of great circles?

Do the same experiments with other kinds of projections.

(ii) Stereographic projection.

We use the same set-up as for central projection, but shine light from North Pole.

(iii) Cylindrical projection from center.

Wrap stiff paper around the sphere to form a cylinder touching the sphere along the equator. Shine light from center. Use translucent paper so one can trace map on the outside of the paper. Then flatten and get a plane map.

(iv) Cylindrical horizontal projection.

Use same set-up as (iii).

For any point on the sphere shine light from the same height as the point (always from line joining the poles).

INTUITIVE WORK INVOLVING THE CONCEPTS OF SYMMETRY, CONGRUENCE AND RIGID MOTION[1]

Before the concepts of congruence, symmetry and rigid motion are studied on a theoretical level children should have a good intuitive grasp of these concepts. On the earliest level children match simple figures made out of cardboard, felt, and paper, for example. Pieces can also be fitted into spaces from which they were cut out. Building with blocks and making copies of their own constructions helps build up the concept of congruence in three dimensions. Tracing figures and printing figures using potatoes can be included. Work with Mirror Cards (see I below) continues to build up acquaintance not only with congruence but also with symmetry. In II (below) work involving rigid motions, congruence and symmetry is discussed. The work described elsewhere in the report dealing with tessellations and constructions strengthens some of these concepts further.

I. Mirror Cards[2]

One way to give children an intuitive feeling for concepts such as congruence and symmetry is by use of the Mirror Cards. The Mirror Cards consist of fourteen different sets of cards. Each card has a pattern on it. Each set has instructions on its cover card, and there is a teacher's guide. The basic problem, however, is the same for all the sets. Can one match a pattern on one card to the pattern on another card by using a combination of some part of that pattern and its reflection in the mirror? The Mirror Cards vary in difficulty - starting with very simple patterns. The approach is non-verbal, the cards are highly visual and they are free of mathematical notation. The children can check their own work without resorting to the authority of the teacher. They can make predictions and immediately check and if necessary amend their predictions. They gain experience in recognizing congruent figures. They gain experience in visualizing figures after they have been reflected in a line. (The light is physically reflected by the surface of the mirror. The resulting image in the plane of the paper is the reflection of the pattern about a line - the line of contact of mirror and paper.) The children notice that a mirror does not carry out a translation. They notice intuitively that congruency of two parts is necessary but not sufficient for a pattern to have been made from a picture and its mirror image. Playing freely with the mirror enables them to watch the change of relative position of image and pattern as the mirror is moved.

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[1] Most of this work is described in more detail in a paper, "Informal Geometry for Young Children", by M. Walter.

[2] The Mirror Cards have been produced in a trial version by E.S.S. A teacher's guide is available. It contains copies of one of the sets.

II. SYMMETRY PROPERTIES OF FIGURES. RECOGNIZING CONGRUENT FIGURES.

This work gives experience:

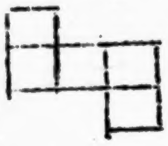
a. In making patterns
b. In manipulating patterns
c. In visualizing patterns
d. With congruency and recognizing it
e. With symmetry and recognizing it

A. Arranging squares

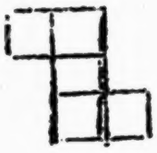
Pick a simple shape such as a box with equal sides and no top. How many sides does it have? How might it look flattened out? How many ways are there of arranging five squares? Obtain all ways of arranging five squares. Which fold into boxes without tops?

B. Congruency:

Decide when two patterns are congruent by actually using paper cutouts and moving one on top of the other. For example,



and

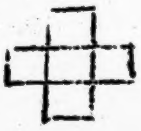


are congruent because one fits on top of the other.

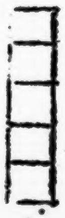
Game: Divide the class into two teams. Each team tries to draw all twelve patterns on the board. This will probably result in several patterns being repeated such as the two figures above.

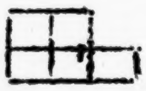
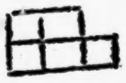
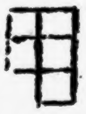
C. Examining the patterns

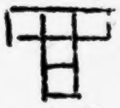
Why are some patterns duplicated more often than others? For example,

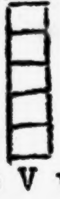


and

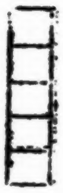


are not often drawn twice, but , for example, may be drawn as  and .

Consider how many different positions each shape has.. For example,  has four different positions (in which edges are vertical or horizontal). Find out that each of these twelve pieces has either 8, 4, 2, or 1 different positions.

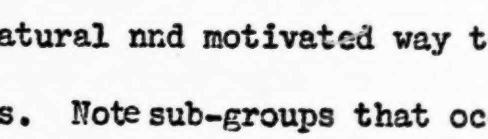
Discuss which motions leave the pattern in the same position. (This work can be extended to include paper folding and mirrors.) For example, for the piece  a half turn 1/2, a full turn 1, a horizontal flip H, a vertical flip V will leave the piece in the same position. What is the result of H 1/2, 1/2 1/2, 1, 1/2 1/2 1, etc.? Students will probably soon be able to calculate the results of HHHHHHHHHH 1/2, HVVVVH, etc.

Can one tell what motion has been made if the piece is not marked and one was not watching the motion? Lead to a discussion of how to mark the figure. Try all the suggestions until an adequate marking is made.

Game: Half the class can have eyes closed. Consider , say.

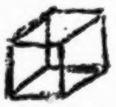
Make a motion such as a 1/2 turn. Let those students who had their eyes closed tell what the motion was. Repeat as often as is of interest. Probably students on their own will give the product of two or more motions. If not, introduce it. Now the game becomes more interesting since there is more than one possibility. What motion is equivalent to the product of two? If students cannot visualize, let them use marked paper cutouts.

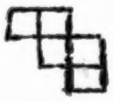
Make a list of the possible successive two motions equivalent to each one motion. This leads the children in a natural and motivated way to make the group table. Repeat with other patterns. Note sub-groups that occur. Are any two groups isomorphic? Repeat with other shapes such as all patterns of four equilateral triangles.

For younger children use a felt board or individual shapes that they can stick together. For example, give each child four equilateral triangles. When he has made a pattern, he can stick it together. He can check whether it is congruent to one he has already made by actually placing the new one on the old one. The younger children can make three dimensional figures from their shapes. Play a game: Move one square on a felt board to a new position to obtain a new pattern. Example: 

D. Extending the work

The work with squares, triangles and other shapes can be extended in several directions. This work might lead to the question of how they could actually construct squares and equilateral triangles. As mentioned earlier, one can raise the question of which patterns fold - and into what shapes. At an earlier age paper cutouts can actually be used. At a later age the students can try to decide by visualizing a pattern drawn. One can ask questions such as: Which square becomes the bottom? Where can one add a square to make a box with a top? In any pattern how many places are there for adding a sixth square? Which are congruent? Given a box without a top, which cuts along edges will flatten it? What does the flattened pattern look like? Example:





From the question of how many ways are there of cutting a figure to flatten it, one may lead to the more general problem of: How many "essentially" different ways are there of choosing 1, 2, 3, etc., edges of a figure - say, a cube.

Another version - perhaps at an earlier level: Take three or four or five sticks. How many different ways are there of joining them if right angled turns are demanded at every joint? (A game based on this and the following ideas can be played.[3]) If straight or right angles are allowed? Other specific angles? This can be asked in two or three dimensions. One may ask: how many ways are there of arranging four or five squares in space if right angled turns are demanded at every edge? How does this compare to arranging four sticks? These problems will entail a discussion of mirror images in three dimensions.

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[3] This is being developed by M. Walter.

E. Other work involving congruency

Use sticks that can be joined at their ends of lengths 3, 4, 5, 6, 7, . . . 10 inches, say. Pick three sticks and make a triangle is possible. Which triangles are congruent? Perhaps discuss similar triangles. Realize S.S.S. condition for congruency. The sum of two sides of a triangle is greater than the third side. Do two sides of a triangle determine it? Get the notion of S.A.S. (this requires further thought and development). Pick four sticks. Make a quadrilateral in the plane or in space. What can be said about congruency? Investigate congruency of other figures.

Consider areas of congruent figures, perimeters. Can two figures which are not congruent have the same areas? Same perimeter? Given a perimeter, make a figure - have children compare their areas. Which shape gives biggest area? Smallest? Make a shape of a given area. How do the perimeters compare?

Section IV - Chapter 17

NOTES TOWARD A DISCUSSION OF TRANSFORMATION GROUPS

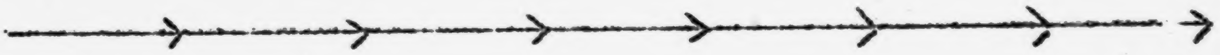
After working with the mirror cards, one discusses what a mirror perpendicular to a plane does to a figure in the plane - and how this result can be gotten by a "rotation" which is called a reflection. Also, what does a mirror do to objects in space?

The symmetries of the figures such as those composed of five squares have already been discussed by the children and some group tables have already been made, and so one may proceed to more formal things.

Consider equilateral triangle. Consider first its rigid motions in the plane, I = identity, R = rotation about center by 1/3 of a circle, and $S = R^2$. Make a table for these operations. Then allow flipping the plane and introduce the three altitudes. Compose the six operators; make a table of their multiplications - practice associative law, inverses, etc.; non-commutativity. Then label the vertices 1, 2, 3, and write each operation as a permutation of vertices $\begin{pmatrix}123\\abc\end{pmatrix}$; then deal with permutations in themselves, count them, make a table of their multiplications. Show that the one-to-one correspondence is an isomorphism (and that we have the symmetric group on three letters).

Then discuss symmetric group on four or more letters, subgroups, n!, etc. (without ever defining a group). This would be useful for probability later. Do the group of rigid motions for the square in the manner used for the triangle; do we get the full symmetric group?

Get the dihedral groups as rigid motions of regular polygons of $n \geq 3$ sides; it generalizes triangle and square done already. Now pass to symmetries of infinite patterns. For example,



which allows an infinite cyclic group of translations combined with the reflection of this line in itself.

Or the figure consisting of a dot for each point of the plane with integral coordinates; or use a tessellation of the plane by regulating hexagons.

The important thing here is for the children to realize that the number of symmetries is ∞ but that there are some simple generators. They might enjoy doing permutations of infinite sets.

The symmetries of solid figures such as the cube and the regular tetrahedron are also worth discussing.

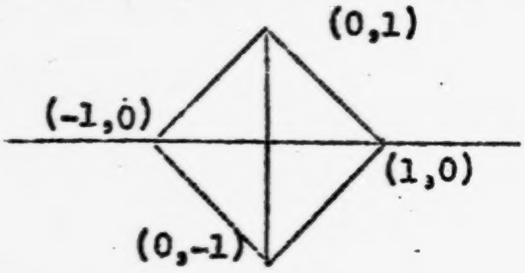
ROTATIONS, MATRICES AND HOMOMORPHISM

Homomorphism, and in particular isomorphism, are of basic importance in modern mathematics. A desirable contribution to the mathematical base of elementary school children would be provided by the introduction of this concept in a way that illustrates its power. Homomorphisms between finite rigid symmetry groups and matrices are useful. The matrices can be set up by direct construction or by using the properties of group structure and homomorphism.

Finite rotation groups have been introduced to third grade. (See Estabrook progress reports for example. A forthcoming C.C.S.M. report will give details of these classes.) They prove to be interesting to the student and feasible for a broad range of abilities. Matrices also have been introduced (as in the Madison Project) motivated by linear operations other than symmetry motions. We propose here, after developing the rotation group structure in two dimensions (with"twists" or inversions) as in the Estabrook project, to build up the 2 x 2 matrix representations, emphasizing the use of homomorphic properties in their construction.

We assume that the children have studied the rigid symmetry motions of regular polygons. The transformation elements have been discussed, together with their equivalence classes, closure, identity and inverse. The students have found that the rotations commute among themselves, but that the twists do not commute with the rotations or with all other twists. This has been done for equilateral triangles and squares in detail, and extended to all regular polygons. In addition the symmetries of a rectangle may have been examined.

Bringing the students' attention back to a square, consider it on Cartesian co-ordinates as in the figure. List the co-ordinates of each vertex before and after a $\frac{\pi}{2}$ rotation.



$(1,0) \rightarrow (0,1) \quad (0,1) \rightarrow (-1,0) \quad (-1,0) \rightarrow (0,-1) \quad (0,-1) \rightarrow (1,0)$

These are "jumps in the plane". If they have had Page's recent material, they will already have arrived at rotations from algebraic expressions. If not, they should now be asked if these jumps can be written in terms of linear equations (of which they should have had previous experience). They may try some of the form $\square^n = \square + a$ $\quad \Delta^n = \Delta + b$ and other inhomogeneous types (we use the superscript n for new co-ordinate). Having solved for a and b, for one vertex, they should be asked to see if it works for another. They will not have success with all their vertices until they try the form,

$$\square^n = a\square + b\Delta$$
$$\Delta^n = c\square + d\Delta$$

Because of the zeros appearing in the co-ordinates of each point they will quickly get $(a,b,c,d) = (0,-1,1,0)$ by using the movement of two vertices. Does it work for all four vertices? What does this transformation do to the co-ordinates of points on the edges of the square? any points? To the origin? The last shows why the transformation is homogeneous.

They should then find (a,b,c,d) for π and $\frac{-\pi}{2}$ rotations, and for twists (or at least for the two easiest twists). As (a,b,c,d) determine the transformation equations above, the teacher can suggest the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a mnemonic for the set of equations. They can then consider successive motions $(\Box, \triangle) \rightarrow (\Box^{n}, \triangle^{n}) \rightarrow (\Box^{nn}, \triangle^{nn})$. What rotation matrix does one end up with following a rotation of $\frac{\pi}{2}$ by one of π; a π rotation by another π rotation? What is the identity matrix?

They should now look at their successive operations without performing the arithmetic at each step. They would then obtain for a rotation $\frac{\pi}{2}$ followed by a π rotation

$$\Box^{nn} = -1 \text{ x } \Box^{n} + 0 \text{ x } \triangle^{n} = -1 \text{ x } [\Box \text{ x } \Box + (1) \text{ x } \triangle] + 0 \text{ x } [1 \text{ x } \Box + 0 \text{ x } \triangle]$$
$$= [-1 \text{ x } 0 + 0 \text{ x } 1] \text{ x } \Box + [-1 \text{ x } (-1) + 0 \text{ x } 0] \text{ x } \triangle$$

and a similar series of steps for $\triangle^{nn}$.

They have already used an operator notation for the geometrical symmetry motions: r^1(polygon) = (rotated polygon), r^2 (r^1 (polygon)) = r^2 (polygon rotated according to r^1), etc. To follow up their algebraic transformations concisely it should be suggested that they write

$$(\Box^{n}, \triangle^{n}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\Box, \triangle) .$$

It is not yet natural to suggest writing the co-ordinates in a column, but it will be later. Now looking at their successive transformations in operator form they have

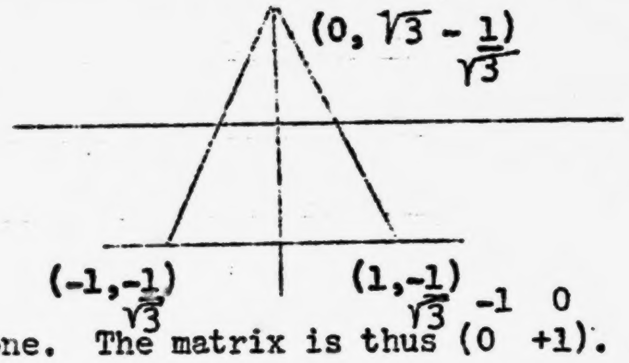
$$(\Box^{nr}, \triangle^{nn}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\Box, \triangle) \right\}$$
$$\begin{pmatrix} -1 \text{ x } 0 + 0 \text{ x } 1 & -1 \text{ x } (-1) + 0 \text{ x } 0 \\ 0 \text{ x } 0 + (-1) \text{ x } 1 & 0 \text{ x } (-1) + (-1) \text{ x } 0 \end{pmatrix} (\Box, \triangle)$$

In this the pattern of row times column can be observed. This should be tested for other successive symmetry motions of the square. Then one can indicate that if one writes $\begin{pmatrix} \Box^{n} \\ \triangle^{n} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Box \\ \triangle \end{pmatrix}$ the same pattern of row times column gives $\begin{pmatrix} \Box^{n} \\ \triangle^{n} \end{pmatrix} = \begin{pmatrix} a\Box + b\triangle \\ c\Box + d\triangle \end{pmatrix}$ from which one can read off the linear algebraic equations.

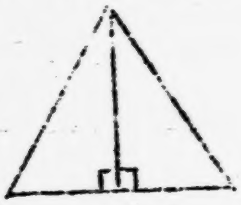
The properties of this multiplication among the set of matrices should now be investigated. It is expected that the student will soon find that he can predict results of matrix multiplication by using the correspondence (homomorphic connection to the symmetry motions and isomorphic to the equivalence classes of symmetry motions) to the geometric results. They should be able to find several ways of generating transformation matrices, including any matrices which they had not previously obtained directly. The question of closure can be demonstrated by calculation, but also by reference to the closure of the geometrical symmetry motions. The identity element will come up in the same context, and can be found from many pairs of matrices. Inverses must exist according to the correspondence, and can be found in this way much more easily than by setting up the equations $MM^{-1} = I$. The question of whether matrices commute under multiplication can be found out by trial, or predicted by correspondence. If addition of matrices has been defined, there may be some amusement in looking at the commutators.

It is proposed that the students now try to find the matrix representation of the symmetry group of the equilateral triangle. This time they are to see how much the construction can be simplified by using the isomorphic correspondence. How many matrices do they have to find directly in order to generate the rest by multiplication?

If the triangle is placed on the co-ordinate system in this way, the symmetry about the vertical axis makes the "vertical twist" matrix easy to find even before the vertex co-ordinates are known. For all three vertices this motion changes the sign of □ and leaves △ alone. The matrix is thus $\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$.


$(0, \sqrt{3} - \frac{1}{\sqrt{3}})$
$(-1, -\frac{1}{\sqrt{3}})$
$(1, -\frac{1}{\sqrt{3}})$

Multiplication of this by itself gives the identity, so we need more. A rotation or a "diagonal twist" must be found, requiring the location of vertices. We can obtain these by finding the ratios of the sides of the 30°, 60°, 90° triangle obtained by splitting the equilateral triangle in half. The 90° angles follow from symmetry. By construction $|ab| = 1/2\ |ac|$. If $|ab|$ is one unit long, then the Pythagorean theorem tells us that $|bc|^2 = 2^2 - 1^2 = 3$. An accurate evaluation of $\sqrt{3}$ is not required for what follows as we only use the fact that $\sqrt{3} \times \sqrt{3} = 3$. The class should however know what it is to about 5%, so that they can recognize gross errors in their drawings and know the sign of $\sqrt{3} - \frac{1}{\sqrt{3}}$. As, for rotation, they want the center of the triangle to be at the origin, they must find $|dc| = |da|$. By symmetry $\angle dab = \angle dac$. It follows that bda is similar to bac, and that $|bd| = \frac{ba}{bc} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$. Then $|dc| = \sqrt{3} - \frac{1}{\sqrt{3}}$ and the co-ordinates of the other vertices are easily found to be as in the figure.



To find the coefficients of the transformation easily one wants to transform a point which has one zero co-ordinate. How do we transform $(0, \sqrt{3} - \frac{1}{\sqrt{3}})$ to $(-1, -\frac{1}{\sqrt{3}})$?

$$-1 = a \times 0 + b \times (\sqrt{3} - \frac{1}{\sqrt{3}})$$

$$-\frac{1}{\sqrt{3}} = c \times 0 + d \times (\sqrt{3} - \frac{1}{\sqrt{3}})$$

The first equation gives $b = -\dfrac{1}{\sqrt{3} - \frac{1}{\sqrt{3}}}$, and the second $d = -\dfrac{\frac{1}{\sqrt{3}}}{\sqrt{3} - \frac{1}{\sqrt{3}}}$.

What if the same transformation is to bring $(1, -\frac{1}{\sqrt{3}})$ to $(0, \sqrt{3} - \frac{1}{\sqrt{3}})$?

$$0 = a \times 1 + b \times (-\frac{1}{\sqrt{3}})$$

$$\sqrt{3} - \frac{1}{\sqrt{3}} = c \times 1 + d \times (-\frac{1}{\sqrt{3}})$$

The first equation gives $a = \frac{1}{\sqrt{3}}\ b = \frac{-1}{\sqrt{3}} / (\sqrt{3} - \frac{1}{\sqrt{3}}) = d$

and the second gives

$$c = \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{1}{3} / (\sqrt{3} - \frac{1}{\sqrt{3}}) = \frac{(\sqrt{3} - \frac{1}{\sqrt{3}})^2 - \frac{1}{3}}{\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{3 - 2}{\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{3} - \frac{1}{\sqrt{3}}} = -b \ .$$

The students can now generate all the remaining matrices by successive multiplications of this and the vertical twist matrix.

The labor of calculating the components of the generating matrices can be reduced by first discovering their orthogonality. Then they only need to solve the first equation above for b, and then they get $c = -b$ and $a = d = \pm\sqrt{1 - b^2}$ from the orthogonality. The sign in front of the radical can be checked by inserting in the second equation. What does the other sign do?

The orthogonality is approached by noting that

$$(\square^n)^2 + (\triangle^n)^2 = (\square)^2 + (\triangle)^2 \qquad \text{or}$$

$$(a^2 + c^2)\square^2 + 2\,(ab + cd)\square\,\triangle + (b^2 + d^2)\triangle^2 = \square^2 + \triangle^2 \ .$$

A judicious selection of a few particular points (with $\square$ or $\triangle$ vanishing for instance), for which the students know that the distance from the origin is unchanged, will quickly give the result $a^2 + c^2 = 1$, $ab + cd = 0$, $b^2 + d^2 = 1$. Insertion of results of the first and third into the second gives $\sqrt{1 - c^2} \times b \pm c\sqrt{1 - b^2} = 0$, which is solved by $c = -b$. That one should choose $a = d$, $c = -b$ for a rotation is indicated by the special case of the identity. The opposite case is an inversion. This orthogonality relation could be tested for the known $\frac{n\pi}{2}$ matrices, and then used to derive the coefficients of the $\frac{2\pi}{3}$ rotation matrix.

They could now look at the matrices representing the symmetry of the rectangle. This is trivially a subgroup of the symmetry group of the square. This relation could be exploited further.

If more practice with matrices, and more insight into the use of the isomorphism is desired, one can continue as follows. What are the matrices corresponding to the symmetry motion of a pentagon? A generating twist is easily obtained if the pentagon is aligned symmetrically with respect to one of the axes. ($a = -d = \pm 1$, $b = c = 0$). A non-trivial rotation is much harder. Can they find a rotation matrix without using geometry or trigonometry? What happens if they multiply the $\frac{2\pi}{5}$ rotation matrix by itself over and over? On equating the fifth power of the matrix to the identity they will get, of course, a nasty equation. They could perhaps solve it numerically to one decimal place. (They know it is between -1 and 1.) They should then check the effect of this approximate matrix on one or more vertices. What is the $\frac{2\pi}{5}$ rotation followed by a twist matrix, approximately? What is the $\frac{4\pi}{5}$ rotation matrix, approximately? If they have had some trigonometry, they could compare these results with $\cos \frac{2\pi}{5}$.

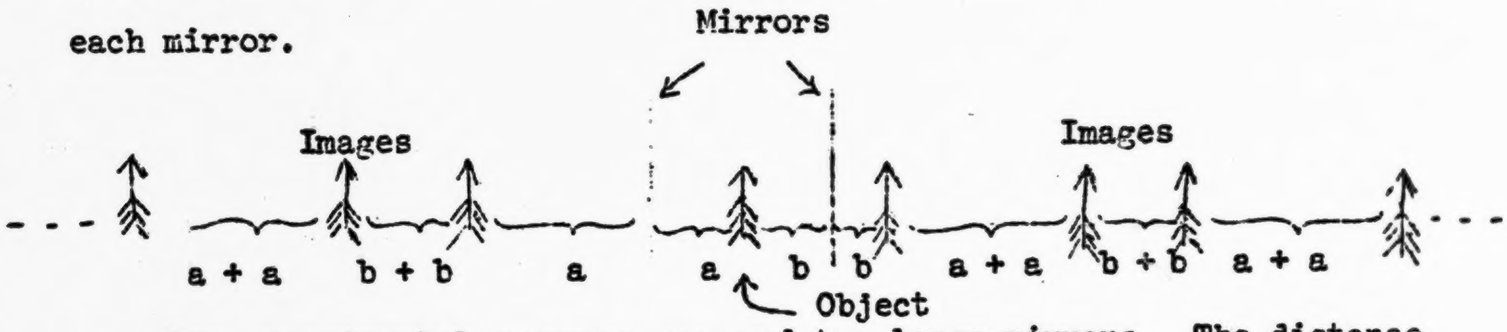
The hexagon has the equilateral triangle and square subgroups. One can then obtain the whole group by combining these and the problem of the polynomial equation does not arise.

Section IV - Chap. 19

ITERATED REFLECTIONS IN MIRRORS

The material given here could be the basis for a presentation in the sixth grade. The students should already know about reflection in single mirrors (having worked with the Walter Mirror Cards, for example). They also should know that the angle sum of a plane triangle is π (or 180° if they don't know radius measure). This material is an introduction to crystallographic groups and is important also for study of Lie groups and Lie algebras. In this sense what the student discovers here is all possible generalized Weyl groups of two-dimensional Lie groups.

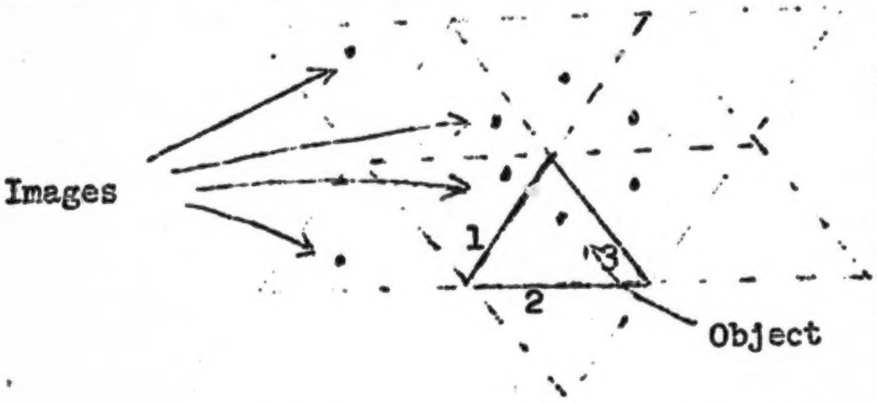
An object has just one image in a mirror, but with two mirrors we may get infinitely-many images. If a room has mirrors on opposite parallel walls (as for example in a barber shop) then one mirror reflects both the object and the reflection of the object in the other mirror. Since both mirrors are doing this we get an infinite sequence of reflections in each mirror.



For experimental purposes we need two large mirrors. The distance arrangement of images should be checked visually for different placements of an object (preferably the observer). In particular we note that the images are all evenly spaced when the object is midway between the mirrors.

Next we observe what happens when the mirrors are not parallel. The more nearly parallel the greater the number of images. Find out at what angle the number of images reduces to two. Try to get more images in one mirror than in the other. These observations should be explained in terms of light rays having equal angles of incidence and reflection.

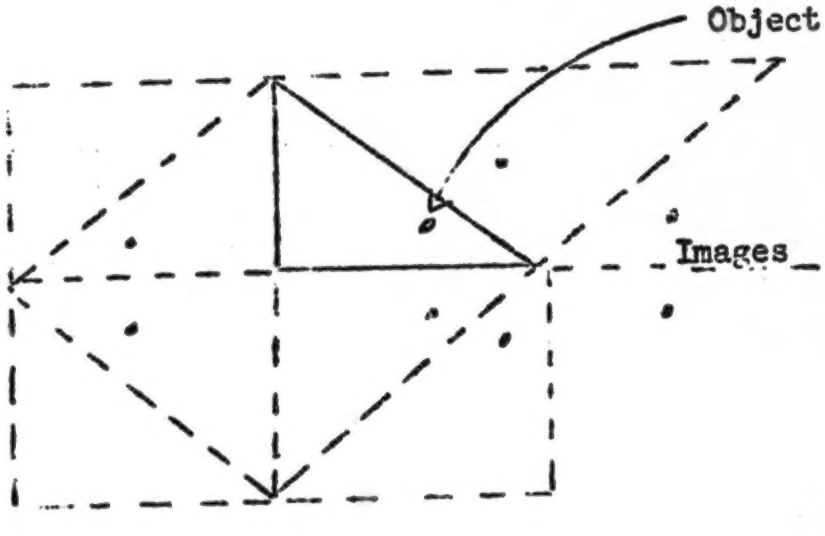
Now we want to use three mirrors so that we can form a triangle with the observer inside. Again we consider the nice situation first by making an equilateral triangle. Here is a view from above:



Solid lines are mirrors, dashed lines are images of mirrors (and images of of images, etc.)

A convenient way to locate images is to use images of the mirrors. Then a new image is obtained by reflecting an image in the image of a mirror. For example, the line labelled a in the diagram above is the image of mirror 1 in mirror 3, and b is the image of mirror 2 in mirror 3. We continue by reflecting images of mirrors in other images of mirrors, etc., and then we reflect object images in mirror images to obtain all object images. Go back and use this kind of construction for two parallel mirrors.

Just as with two parallel mirrors, it is instructive to see what happens when the angles between mirrors are changed (so that the triangle is no longer equilateral). Consider for example a triangle with angles 90°,45°,45°.



As before, the images are evenly spaced throughout the plane in the diagram above. The same situation occurs for a 30°, 60°, 90° triangle.

Now make the mirrors into some triangle other than the special three (equilateral, 45°,45°,90° and 30°,60°,90°). We see that the images are not evenly spaced but tend to "bunch up" in some directions. Let us agree to call the set of images discrete if they stay equally spaced throughout the plane and non-discrete if they bunch up in some directions.

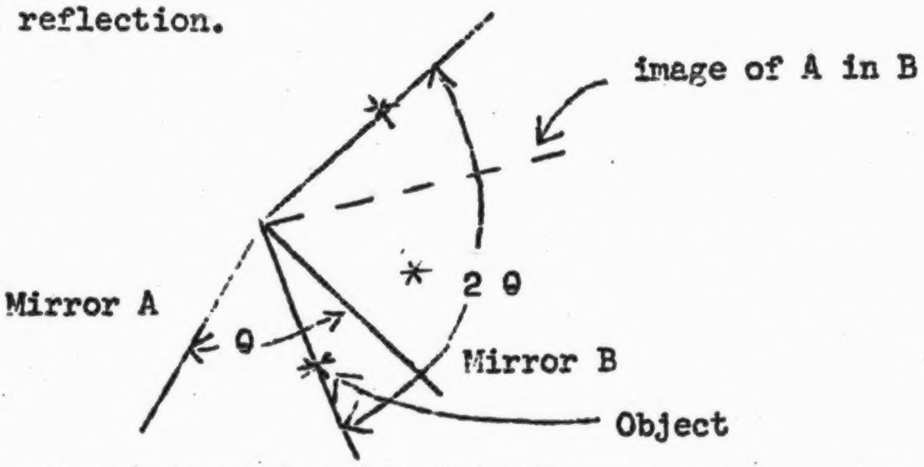
There are two important things to notice.

(1) Whether the images are discrete depends only on which triangle of mirrors we are using and not on the placement of our object in the triangle.

(2) The only triangles which give discrete sets of images are the three special triangles already examined.

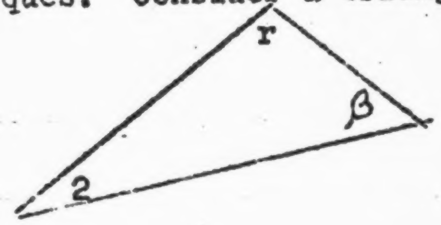
These should be checked by experiment. There is, however, a very pretty proof of (2) and we give a sketch of this proof now.

We must consider what happens when we have two mirrors at an angle and consider an iterated reflection.



It turns out that we can get from the object to the second image by a rotation through an angle 2θ (where the θ is the angle between the two mirrors). That is, an iterated reflection is a rotation through twice the angle between the mirrors. Thus by repeating iterated reflections the image gets rotated around the origin (where mirrors come together), and the image will get back to the object position only if some integral multiple of 2θ is 2π. If this doesn't turn out to be true, then the images will eventually bunch up. (A rigorous proof of this is too difficult to give here.)

Once we accept the statements above, we can complete the proof by more analytical techniques. Consider a triangle with angles $2, \beta$, r



and suppose the iterated images are not to bunch up. Then

(i) $p = \frac{2\pi}{2}$, $q = \frac{\pi}{\beta}$, $r = \frac{\pi}{r}$

must be integers. Also we have

(ii) $2 < \pi$, $\beta < \pi$, $r < \pi$, $2 + \beta + r = \pi$.

Combining (i) and (ii) we get

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi \text{, or}$$

(iii) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

It follows that:

(a) $p, q, r \leq 2$, and at most one of them can equal 2.

(b) We cannot have p, q, r, all > 3.

Using these results we easily get that the only possibilities are that p, q, r are (in some order)

2, 4, 4
2, 3, 6
3, 3, 3

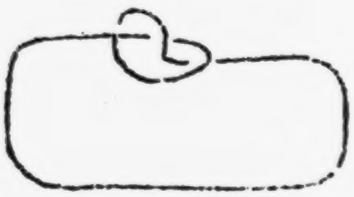
and this proves that triangles with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$ and $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$ and $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$ are the only ones which will give discrete sets of images.

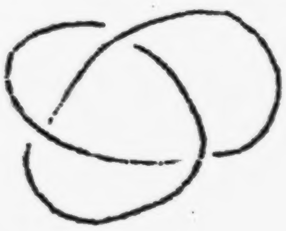
KNOTS

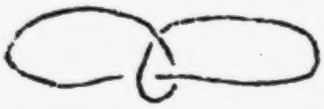
This section should be done when the students are old enough to manipulate rope effectively and to draw representations of knots showing one line passing under another. The equipment used could be lengths (about four feet) of sash cord with some device for attaching the ends together. Probably some kind of sleeve exists or can be devised. Otherwise Scotch tape could be used.

I. Identifying knots.

Tie an overhand knot in a cord and then attach the ends. Put the cord on a table in various positions, e.g.







Do the same for other simple knots.

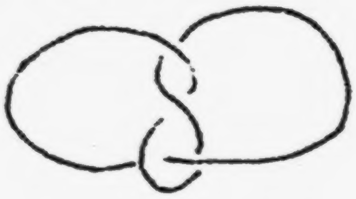
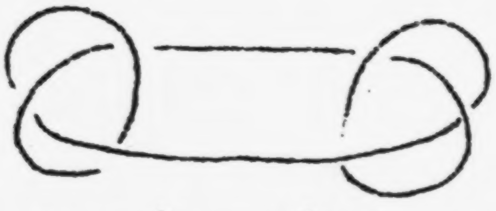
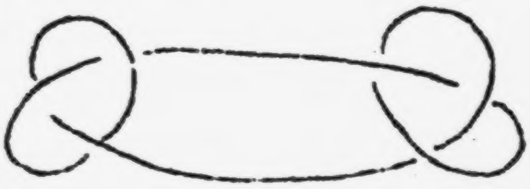


Figure 8 knot



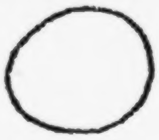
Square(or reef)knot



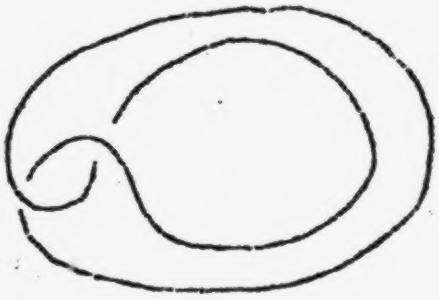
Granny(or false reef)knot

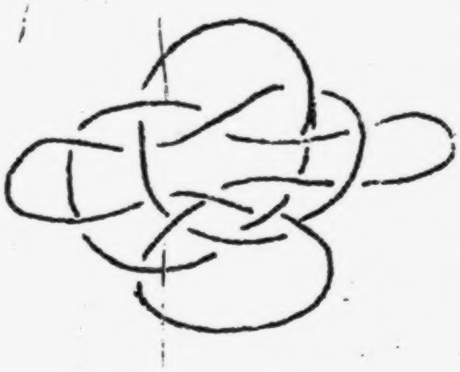
Learn to tie knots from looking at knot diagrams (like those we have just drawn). Learn to draw diagrams from a given knot in a cord.

Given two knotted cords, agree that they are the same knot if there is one diagram which will give both. One can also"put one knotted cord on top of the other" to see if two knots are the same. A cord with the knot diagram



is called unknotted (or is called the trivial knot). Find lots of diagrams representing the trivial knot, e.g.

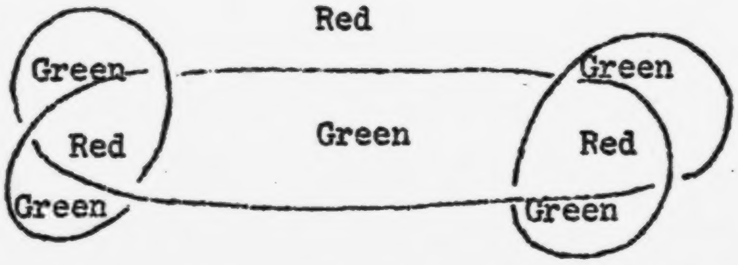




II. Two-color theorem.

We can consider that a knot diagram divides the plane into regions. Try coloring the regions with two colors so that no two regions with a common boundary line have the same color.

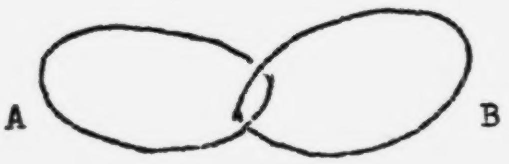
Example



Do this for enough knot diagrams to become convinced that it always works.

III. Links.

Use unknotted cord loops. Two such are linked if they won't come apart. For example, loops A, B are linked:



Section IV- Chap. 21

SPHERE, CYLINDER, TORUS, ETC.

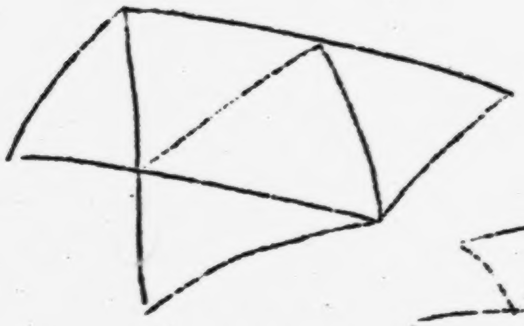
The main point is to study the behavior of curves (in particular geodesics) on various kinds of surfaces.

I. Division of surfaces into triangles.

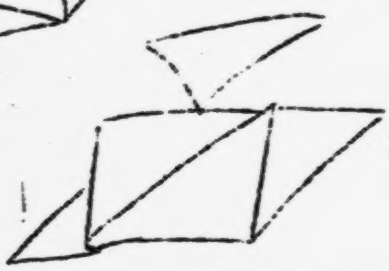
Models of spheres, cylinders and torus are needed on which one may draw and erase lines.

Divide these into curved triangles so that if two triangles have any part in common, it is either a vertex or an edge.

For example,



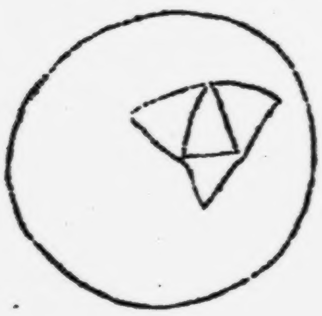
is O.K. but

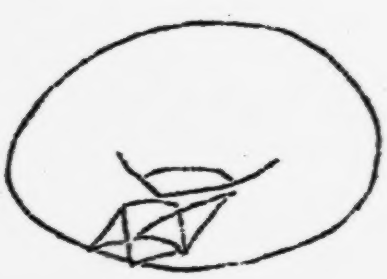


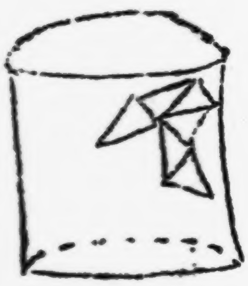
is not.

The triangles are to cover the entire surface. (For the cylinder, the top and bottom edges will have to be made up of edges of triangles.)

For example,







and then continue to fill up surface with triangles.

Count vertices, edges, faces for each. Check that

	Sphere	Torus	Cylinder
V - E + F	2	0	0

Do this several different ways and check that each time the same number is obtained for V - E + F.

II. "Straight lines" on surfaces.

Imagine 2-dimensional creatures living on the surfaces (sphere, torus, cylinder). Ask what would "straight line" mean to them. Evolve notion that taut string held against surface is best answer for sphere and cylinder. For torus it is hard to keep string taut and still on surface for nearby points on the "inner part" but O.K. for outer part. To solve this problem we use narrow strips of colored cellophane tape which will stick.

Give a position (point) on the surface from which to start and a direction at that point. See that by keeping tape tight, allowing no bends, and attaching a small bit at a time, the whole curve is dictated by initial position and direction. It should be interesting to note how small errors build up. Let students start with some initial point and direction and compare end position for fixed (equal) lengths of tape.

Call these curves geodesics.

Show that: Geodesics on the sphere are great circles.

Geodesics on the cylinder are helices (or degenerate helices: a horizontal circle or a vertical line.)

Geodesics on a torus come in several types. (Note: The situation is very complicated and one should not expect these experiments to yield a complete solution.

III. Triangles on surfaces.

Instead of covering surfaces with triangles, we now look at single triangles on surfaces. We consider geodesic triangles: i.e. triangles whose edges are geodesics.

Construct such a triangle on a cylinder. Measure the angles and add them to see that the sum is 180°, just as on a plane. Do this several times.

Cut the cylinder and flatten it to see that we then get triangles made of straight lines in the plane and with the same angles.

Next we construct geodesic triangles on the sphere and find the sum of the angles of each. Check that the answer is always $>180°$. (Don't construct degenerate triangles with two sides coinciding, etc.)

Now we try the same on a torus. Can we get a triangle with angle sum $>$ 180°? with $<180°$?

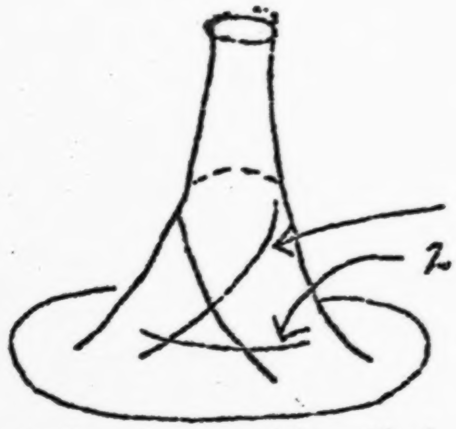
IV. The pseudosphere.

A sphere is a surface with constant positive curvature; so a surface with constant negative curvature is called a pseudosphere. (The curvature of a plane and a cylinder are zero. The curvature of a torus is not constant but varies from point to point.)

Construct (or find some facsimile, e.g. the bell of a trombone) a pseudosphere.

Examine the behavior of geodesics on the pseudosphere, e.g. these two:

1 and 2



Measure the angle sum of triangles, noting that it always comes out < 180°.

This leads to a feeling for curvature of a surface. If we want to know the curvature of a surface at a point, we draw a small geodesic triangle near the point and measure the angle sum. The curvature is defined to be positive if this is > 180°, zero if = 180° and negative if < 180°.

By a limiting process this technique can be used to assign a numerical value to the curvature at each point of a surface.

Concluding Remarks:

It is clear to us that this report represents only a beginning in the development of material that will teach mathematics in the schools as we would like to see it taught. It touches on only a fraction (though not a small one) of the topics for the elementary grades and has not yet peered into the high school period. The presentation contains so little detail and is so unstructured, that it requires mathematicians with experience in curriculum research to attempt its teaching. Much of it has not been tested at any level and so may easily fall on its face. Indeed we have our own grave doubts of the efficacy of much of our material. The question of preparing teachers to use such material has not been raised at this stage.

However, we hope the content of this report is sufficient to allow these further steps to proceed. We hope that its intent and detail are clear enough, and interesting enough, for research people working with children to try them out. This may lead, after a while, to sufficient information about what children can do in elementary school so that one can begin to elaborate the form of the high school mathematics. Every experiment in the classroom gives some suggestion as to the scope of the teacher training problem ultimately involved.

We expect that a series of workshops, interleaved by classroom experience, will prove necessary in an iterated approach to our goals. We will be pleased if our summer's work is of some significance in establishing this chain of development.

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	Unit 17	Measurement	
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	Unit 13	Metric Geometry	
Primary 3	Unit 4	Geometry	
	Unit 8	Measurement	
	Unit 11	Geometry	
Secondary 1	Chapter 5	Sets of Points	
	Chapter 9	Measurements	
Secondary 2	Unit 2	Chapter 8	Use of Instruments
		Chapter 9	Congruence
		Chapter 10	Basic Constructions
		Chapter 11	Symmetry
		Chapter 12	Angle Properties of Circles
		Chapter 13	Loci and Envelopes
		Chapter 14	The Pythagorean Property
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Unit VII	point, line segment, etc. (not seen)
Unit IX	reviews earlier units of geometry - intersecting regions common boundaries, etc.
Unit X	measurement
Unit XIII	geometry - line segments, points, vertices, tiles, measuring regions, triangles, rectangles and squares.
Unit XIV	symmetry

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T, Lengths, Area, Volume
W, Mapping

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	Part 3	Ch. X	Linear Measurement
Book 2	Part 1	Ch. III	Sets of Points
	Part 2	Ch. V	Linear Measurement
	Part 3	Ch. VII	Congruence of Angles and Triangles
Book 3	Part 1	Ch. I	Sets of Points
		Ch. III	Describing points as numbers
	Part 2	Ch. VI	Length and area
Grade 4			Sets of Points
			Recognition of Common Geometric Figures
			Linear Measurement
Grade 5			Congruence of Geometric Figures
			Measurement of Angle
			Area

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